
INSTRUCTIONS: Computers, calculators, books, notes, flailing monkeys, *etc.* are not permitted. Some (possibly) useful formulae are attached. Write your name, your instructor's name and your recitation section (and/or TA's name) on the front of your bluebook. Work all problems. Start each problem on a **new page**. Unless otherwise stated, show **all** your work clearly and $\boxed{\text{box}}$ your final answer; a correct answer with incorrect or no supporting work will **receive no credit**, while an incorrect answer with relevant work may receive partial credit.

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1. (25 points) State whether the following statements are (always) TRUE or FALSE (not always true), for twice continuously differentiable functions $f, \mathbf{F}$. You MUST write the full word "TRUE" or "FALSE" — "T"/"F" will NOT be graded. You do NOT need to show your working or reasoning for this question.

   (a) $\nabla \cdot (\nabla f) = 0$.

   (b) $\nabla \times (\nabla f) = \mathbf{0}$.

   (c) $\nabla \cdot (\nabla \times \mathbf{F}) = \nabla \times (\nabla \cdot \mathbf{F})$.

   (d) If $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$ then $\mathbf{F} = \mathbf{0}$.

   (e) $\nabla \cdot (\nabla f \times \mathbf{F}) + \nabla \cdot (\mathbf{F} \times \nabla f) = 0$.

2. (35 points)

   (a) Calculate the (counterclockwise) circulation of $\mathbf{F} = \dfrac{-y}{x^2+y^2}\hat{\imath} + \dfrac{x}{x^2+y^2}\hat{\jmath}$ around a circle of radius a (centered at the origin). Clearly state any theorems you use and justify their use.

   (b) Calculate the (outward) flux of $\mathbf{F} = \log(x^2+y^2)\hat{\imath} - \log(4-x^2)\hat{\jmath}$ through the path from $(1,1)$ down the line $x = 1$ to $(1,-1)$ then back to $(1,1)$ around the right-hand portion of the circle $x^2 + y^2 = 2$. Clearly state any theorems you use and justify their use.

3. (35 points) The ideal gas law $PV = nRT$ relates the pressure of a gas P to the volume V, the temperature T, and the number of moles of gas n; R is a known constant.

   (a) A section of a bouncy-castle has a volume of 2 m^3 and is filled with gas at a temperature of 300 K and a pressure of 2×10^5 Pa. Is the pressure more sensitive to changes in temperature or to changes in volume? (Assume n and R are fixed.)

   (b) Calculus students celebrating the end of the semester jump on the bouncy-castle, compressing the volume to 1.8 m^3; the beautiful spring weather has also increased the temperature to 303 K. Estimate the change in the pressure. [Note: although you do not know n and R, you can determine the value of nR.]

(c) The calculus students are trying to guess how many moles of gas are in the bouncy-castle. To determine who has guessed correctly, they will measure the volume, pressure and temperature of the gas and calculate n using the ideal gas law. However, each of these measurements will have a small error associated with them. Assuming R is a known constant, show that the percentage error in the calculated value of n is the sum of the percentage errors in V, P and T.

4. (30 points) Calculate the flow of a particle traveling from $t = 0$ to $t = 3\pi$ along the path $\mathbf{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath} + (t/\pi)^2\hat{k}$ in the velocity field $\mathbf{u} = z\hat{\imath} - z\hat{\jmath} + (x - y)\hat{k}$.

5. (35 points) Find the ratio H/R of height to radius that minimizes the total surface area of a right-circular cylinder of fixed volume V.

6. (40 points) Due to an intergalactic shortage of dilithium crystals, the Enterprise has been refitted to replace its warp drives with ionic neutrino flux infinitesimal net internal temperature yield (INFINITY) drives, which are a bit like flux capacitors but more efficient. The Enterprise is currently locked in deadly combat and Cpt. Kirk is, of course, demanding more power; Scotty would like to tell him that he's already giving it everything they've got and the engines cannae take no more, but with this newfangled technology, he's just not sure if that's true. As Engineer's Apprentice, it is your job to do the necessary calculations. (Don't screw up or you'll be demoted to Anonymous Ensign Who's First Through The Door, and we know what always happens to them. . .)

   The INFINITY drive can be described as the region between the paraboloid S, given by $x = 4-y^2-z^2$, and the plane P, given by $x = 0$. The ionic neutrino field generators are pumping out a field $\mathbf{F} = \cos(x)\hat{\imath} + xz\hat{\jmath} + z\sin(x)\hat{k}$. To calculate the power being generated by the engines, Scotty needs you to calculate the (outward) flux of the ionic neutrino field through the surface S.

   (a) Being smart, you decide to make the ship's computer do the donkey work. Set up the integral necessary to calculate the flux directly. In order for the computer to process your request, you need to give the integral set up completely in the most logical coordinate system (*i.e.* do everything except actually evaluate the integral).

   (b) Unfortunately, the computer is already somewhat overloaded with complex navigation calculations, so you will have to do the work yourself. Use your knowledge of Calc III to show how the flux through S can be related to the flux through P. [Hint: S and P together form the outer shell of the solid drives.]

   (c) Determine the flux through S, preferably before you get boarded by Klingons.

7. (0 points) Why did the chicken cross the Möbius Strip?

— Useful and interesting formulae —

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\vec{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\vec{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(u)|\,du \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{[\dot{x}^2+\dot{y}^2]^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \qquad \text{where} \qquad a_T = \frac{d}{dt}|\mathbf{v}|, \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

$$\frac{df}{ds} = D_{\mathbf{u}}f = (\nabla f)\cdot\mathbf{u}$$

$$\text{Discriminant: } \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$$

$$\nabla f = \lambda\nabla g \qquad g = 0$$

$$\begin{aligned} f(x,y) &= f(0,0) + \left(\left.\frac{\partial f}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial f}{\partial y}\right|_{(0,0)} y\right) \\ &\quad + \frac{1}{2}\left(\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} x^2 + 2\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(0,0)} xy + \left.\frac{\partial f}{\partial y^2}\right|_{(0,0)} y^2\right) \\ &\quad + \ldots + \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n\bigg|_{(0,0)} f + \ldots \end{aligned}$$

$$|E(x,y)| \le \frac{M}{2}\left(|x-x_0| + |y-y_0|\right)^2 \qquad \text{where} \qquad |f_{xx}|, |f_{xy}|, |f_{yy}| \le M$$

$$M = \iint_R \delta(x,y)\,dA \qquad M_x = \iint_R y\delta(x,y)\,dA \qquad M_y = \iint_R x\delta(x,y)\,dA$$

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

$$I_x = \iint_R y^2\delta(x,y)\,dA \qquad I_y = \iint_R x^2\delta(x,y)\,dA \qquad I_0 = \iint_R (x^2+y^2)\delta(x,y)\,dA$$

$$R_x = \sqrt{I_x/M} \qquad R_y = \sqrt{I_y/M}$$

$$\begin{aligned} x &= r\cos(\theta) & r &= \rho\sin(\phi) & x &= \rho\sin(\phi)\cos(\theta) \\ y &= r\sin(\theta) & z &= \rho\cos(\phi) & y &= \rho\sin(\phi)\sin(\theta) \\ z &= z & \theta &= \theta & z &= \rho\cos(\phi) \end{aligned}$$

$$\rho = \sqrt{x^2+y^2+z^2} \qquad r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}(y/x) \qquad \phi = \tan^{-1}(r/z)$$

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

$$\begin{aligned} \iiint_V f(x,y,z)\,dx\,dy\,dz &= \iiint_{V'} f(x(u,v,w),y(u,v,w),z(u,v,w))|J(u,v,w)|\,du\,dv\,dw \\ J(u,v,w) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \end{aligned}$$

$$\begin{aligned} \text{Work: } \int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy \\ \text{Flux: } \int_C \mathbf{F}\cdot\mathbf{n}\,ds &= \int_C M\,dy - N\,dx \end{aligned}$$

$$\text{Conservative field: } \nabla\times\mathbf{F} = \mathbf{0} \Leftrightarrow \begin{cases} \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \\ \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z} \\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \end{cases}$$

$$\text{For a surface given by } g(x,y,z)=0: \quad \iint_S f(x,y,z)d\sigma = \iint_R f(x,y,z)\frac{|\nabla g|}{|\nabla g\cdot\mathbf{p}|}\,dA$$

$$\text{Flux of } \mathbf{F} \text{ through surface given by } g(x,y,z)=0: \quad \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R \frac{\pm\mathbf{F}\cdot\nabla g}{|\nabla g\cdot\mathbf{p}|}\,dA$$

Green's Theorem:

$$\begin{aligned} \text{Circulation} &= \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R (\nabla\times\mathbf{F})\cdot\hat{k}\,dA = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\ \text{Flux} &= \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \nabla\cdot\mathbf{F}\,dA = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA \end{aligned}$$

Stokes's Theorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C M dx + N dy + P dz = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma$$

Divergence Theorem of Gauss:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV$$