APPM 2350: Final Exam (6 May 2015)

Prob 1)

a) $\nabla\cdot(\nabla f)=0$	False
b) $\nabla\times(\nabla f)=0$	True
c) $\nabla\cdot(\nabla\times\vec{F}) = \nabla\times(\nabla\cdot\vec{F})$	False
d) If $\nabla\cdot\vec{F}=0$ & $\nabla\times\vec{F}=0$ then $\vec{F}=0$	False
e) $\nabla\cdot(\nabla f\times\vec{F})+\nabla\cdot(\vec{F}\times\nabla f)=0$	True

Prob 2)

a) $\vec{F} = -\dfrac{y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$ $\qquad C: x^2+y^2=a^2$

$\vec{F}$ is discontinuous at the origin, so $\nabla\times\vec{F}\neq 0$ at the origin (not differentiable). So do the path integral directly.

Let $x = a\cos\theta \Rightarrow dx = -a\sin\theta\, d\theta$

$y = a\sin\theta \Rightarrow dy = a\cos\theta\, d\theta$

$0\le\theta\le 2\pi$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\left(-\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}\right)\cdot\left(-a\sin\theta\,\hat{i}+a\cos\theta\right)d\theta$$

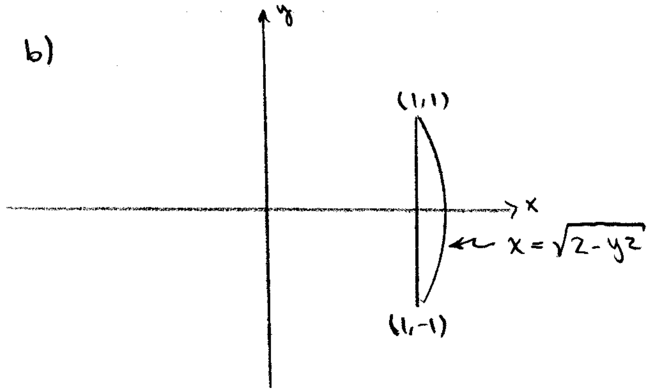
$$= \int_0^{2\pi}\left(-\frac{a\sin\theta}{a^2}\hat{i} + \frac{a\cos\theta}{a^2}\hat{j}\right)\cdot\left(-a\sin\theta\,\hat{i}+a\cos\theta\right)d\theta$$

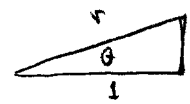
$$= \int_0^{2\pi}(\sin^2\theta+\cos^2\theta)\,d\theta$$

$$= \int_0^{2\pi} d\theta$$

$$\boxed{= 2\pi}$$

b)





$r = \sec\theta$

$\vec{F} = \log(x^2+y^2)\hat{\imath} - \log(4-x^2)\hat{\jmath}$

Flux in the plane $= \oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$ 　　Green's thm in the plane (Divergence form)

$$= \iint_R \frac{2x}{x^2+y^2}\,dA$$

$$= \int_{-\pi/4}^{\pi/4} \int_{\sec\theta}^{\sqrt{2}} \frac{2r\cos\theta}{r^2}\, r\,dr\,d\theta$$

$$= 2\int_{-\pi/4}^{\pi/4} \int_{\sec\theta}^{\sqrt{2}} \cos\theta\,dr\,d\theta$$

$$= 2\int_{-\pi/4}^{\pi/4} (\sqrt{2}\cos\theta - 1)\,d\theta$$

$$= 2\sqrt{2}\sin\theta \Big|_{-\pi/4}^{\pi/4} - \pi$$

$$\boxed{= 4 - \pi}$$

Prob 3)

$$PV = nRT$$

a) $V = 2\,m^3$

$T = 300\,k$

$P = 2\times10^5\ Pa$

$$P = \frac{nRT}{V} \Rightarrow dP = nR\left(\frac{dT}{V} - \frac{T}{V^2}dV\right)$$

$$= nR\left(\frac{1}{2}dT - \frac{300}{4}dV\right)$$

$\boxed{\text{More sensitive to changes in volume}}$

b) $dV = -0.2\,m^3$

$dT = 3^\circ k$

$$PV = nRT \Rightarrow nR = \frac{PV}{T}$$

$$\Rightarrow dP = \frac{PV}{T}\left(\frac{dT}{V} - \frac{T}{V^2}dV\right)$$

$$= \frac{P}{T}dT - \frac{P}{V}dV$$

$$= 2\times10^5\left(\frac{3}{300} + \frac{0.2}{2}\right)\ Pa$$

$$= 2.2\times10^4\ Pa$$

c) $n = \dfrac{PV}{RT}$

$$\Rightarrow dn = \frac{V}{RT}\,dP + \frac{P}{RT}\,dV - \frac{PV}{RT^2}\,dT$$

$$\boxed{\begin{aligned}\Rightarrow \frac{dn}{n} &= \frac{RT}{PV}\left(\frac{V}{RT}\,dP + \frac{P}{RT}\,dV - \frac{PV}{RT^2}\,dT\right)\\ &= \frac{dP}{P} + \frac{dV}{V} - \frac{dT}{T}\end{aligned}}$$

Prob 4)

$0 \le t \le 3\pi$

$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + (t/\pi)^2\,\hat{k}$

$\vec{u} = z\hat{i} - z\hat{j} + (x-y)\hat{k}$

$$\nabla \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -z & x-y \end{vmatrix} = \hat{i}(-1+1) + \hat{j}(1-1) + \hat{k}(0) = 0$$

So $\vec{u}$ is conservative and hence $\int_C \vec{u} \cdot d\vec{r}$ is independent of the path

Now, $\vec{r}(0) = \hat{i}$

$$\vec{r}(3\pi) = \cos 3\pi\,\hat{i} + \sin 3\pi\,\hat{j} + 9\hat{k}$$

$$= -\hat{i} + 9\hat{k}$$

Consider the alternate path $C_1 + C_2$ where:

$$C_1: \quad \vec{r}(\gamma) = (1-\gamma)\hat{\imath} \quad \Rightarrow \quad d\vec{r}(\gamma) = -d\gamma\,\hat{\imath} \qquad 0 \le \gamma \le 1$$

$$C_2: \quad \vec{r}(\gamma) = -\hat{\imath} + \gamma\hat{k} \quad \Rightarrow \quad d\vec{r}(\gamma) = d\gamma\,\hat{k} \qquad 0 \le \gamma \le 9$$

Then

$$\int_C \vec{u}\cdot d\vec{r} = \int_{C_1} \vec{u}\cdot dr + \int_{C_2} \vec{u}\cdot d\vec{r}$$

$$= -\int_0^1 0\,d\gamma + \int_0^9 -d\gamma$$

$$= -9$$

Prob 5)

$$A = 2\pi RH + 2\pi R^2$$

$$V = \pi R^2 H = \text{constant}$$

$$\nabla A = \lambda \nabla V$$

$$\Rightarrow (2\pi H + 4\pi R)\hat{i} + 2\pi R\,\hat{j} = \lambda\, 2\pi RH\hat{i} + \lambda \pi R^2 \hat{j}$$

$$\Rightarrow \quad \cancel{2\pi} H + \overset{2}{\cancel{4}}\pi R = \lambda\, \cancel{2\pi} RH$$

$$2\cancel{\pi} R = \lambda \cancel{\pi} R^2$$

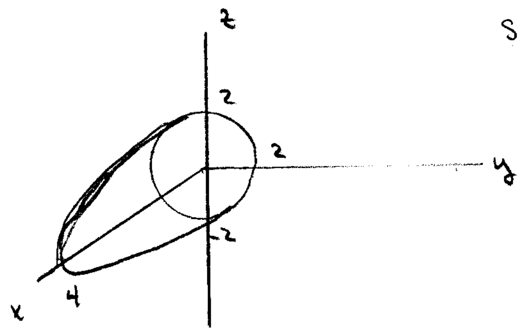
$$\frac{H + 2R}{RH} = \frac{2}{R}$$

$$\Rightarrow H + 2R = 2H$$

$$\Rightarrow 2R = H$$

$$\boxed{\Rightarrow \frac{H}{R} = 2}$$

Prob 6)



S: $x = 4 - y^2 - z^2$

$x = 0$

$\vec{F} = \cos x\,\hat{i} + xz\,\hat{j} + z\sin x\,\hat{k}$

a) $\displaystyle\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_{\text{paraboloid}} \vec{F}\cdot\hat{n}\,d\sigma$

For the paraboloid: $f = x + y^2 + z^2 - 4$

$\Rightarrow \nabla f = \hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$

and thus:

$$\iint_{\text{paraboloid}} \vec{F}\cdot\hat{n}\,d\sigma = \iint_{\text{circle}} \left(\vec{F}\cdot\frac{\nabla f}{|\nabla f|}\right)\frac{dA\;|\nabla f|}{|\hat{i}\cdot\nabla f|}$$

$$= \iint_{y^2+z^2\le 4} \left(\vec{F}\cdot\frac{\vec{\nabla f}}{|\hat{i}\cdot\nabla f|}\right) dA$$

$$= \iint_{y^2+z^2\le 4} \vec{F}\cdot\frac{\hat{i} + 2y\hat{j} + 2z\hat{k}}{1}\,dA$$

$$= \iint_{\substack{y^2+z^2\le 4\\ x = 4-y^2-z^2}} \cos x + 2xyz + 2z^2\sin x\;dA$$

Hence:

$$\boxed{\text{Flux through } S = \int_0^{2\pi}\int_0^{2}\left[\cos(4-r^2) + 2(4-r^2)r^2\cos\theta\sin\theta + 2r^2\sin\theta\sin(4-r^2)\right] r\,dr\,d\theta}$$

b) $$\oiint_{S\cup P} \vec{F}\cdot\hat{n}\,d\sigma = \iiint_V (\nabla\cdot\vec{F})\,dV$$ from the divergence theorem

Further:

$$\oiint_{S\cup P} \vec{F}\cdot\hat{n}\,d\sigma = \iint_S \vec{F}\cdot\hat{n}\,d\sigma + \iint_P \vec{F}\cdot\hat{n}\,d\sigma$$

$$\boxed{\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iiint_V (\nabla\cdot\vec{F})\,dV - \iint_P \vec{F}\cdot\hat{n}\,d\sigma}$$

c) $\nabla\cdot\vec{F} = -\sin x + \sin x = 0 \Rightarrow \iiint_V (\nabla\cdot\vec{F})\,dV = 0$

$$\iint_P \vec{F}\cdot\hat{n}\,d\sigma = -\iint_{\substack{y^2+z^2\le 4\\ x=0}} \vec{F}\cdot\hat{\imath}\,dA$$

$$= -\iint_{y^2+z^2\le 4} dA$$

$$= -4\pi$$

$$\boxed{\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,d\sigma = 4\pi}$$