1 (a) $\vec{AB} = 2\hat{\imath} - 2\hat{\jmath}$

$\vec{AC} = 2\hat{\imath} - 2\hat{k}$

$$\Rightarrow \vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{vmatrix} = 4\hat{\imath} + 4\hat{\jmath} + 4\hat{k} = 4(\hat{\imath} + \hat{\jmath} + \hat{k})$$

$\Rightarrow$ plane is $x + y + z = D$

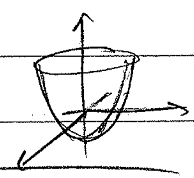
Goes through $(-1,1,1) \Rightarrow -1 + 1 + 1 = D \Rightarrow D = 1$

$\Rightarrow \boxed{x + y + z = 1}$

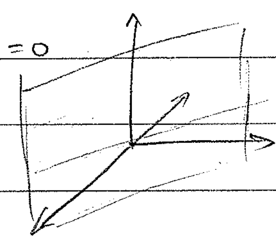
(b) $\vec{PA} = 2\hat{\jmath} + \hat{k} \Rightarrow d = \left| \vec{PA} \cdot \frac{\vec{n}}{|\vec{n}|} \right| = \left| (2\hat{\jmath} + \hat{k}) \cdot \frac{(\hat{\imath} + \hat{\jmath} + \hat{k})}{\sqrt{3}} \right|$

$= \left| \frac{3}{\sqrt{3}} \right| \Rightarrow \boxed{d = \sqrt{3}}$

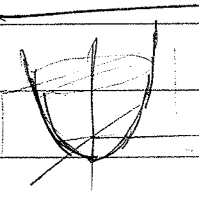
(c) $z = x^2 + y^2 - 1 \rightarrow$ paraboloid



plane $x + y = 0$



$\Rightarrow$ Intersection



parabola

(d) Let $x(t) = t$ Then $y = -x \Rightarrow y = -t$ & $z = x^2 + y^2 - 1$

$= t^2 + (-t)^2 - 1$

$\Rightarrow \boxed{\vec{r}(t) = t\,\hat{\imath} - t\,\hat{\jmath} + (2t^2 - 1)\,\hat{k}}$

2 a) $\boxed{\text{FALSE}}$ e.g. $\hat{\imath}\cdot\hat{\jmath} = 0 = \hat{k}\cdot\hat{\jmath}$ but $\hat{\imath} \neq \hat{k}$

b) $\boxed{\text{TRUE}}$ By definition, $\vec{A}\times\vec{B}$ is normal to $\vec{A}$ ($\&\ \vec{B}$)

Since $\vec{A} \perp (\vec{A}\times\vec{B})$, $(\vec{A}\times\vec{B})\cdot\vec{A} = 0$

c) $\boxed{\text{FALSE}}$ Since $|\vec{v}| = \text{const.}$ $a_T = \frac{d}{dt}|\vec{v}| = 0$ & $a_N = \kappa|\vec{v}|^2$

$= (\text{const.})\,\kappa$

However κ is not necessarily constant

$$\kappa = \left|\frac{d\hat{T}}{ds}\right| = \left|\frac{d\hat{T}}{dt}\right|\frac{1}{|\vec{v}|}$$

$$= \left|\frac{d\vec{v}}{dt}\right|\frac{1}{|\vec{v}|^2} \qquad \left(\hat{T} = \frac{\vec{v}}{|\vec{v}|} \Rightarrow \frac{d\hat{T}}{dt} = \frac{d\vec{v}/dt}{|\vec{v}|}\right)$$

$$= |\vec{a}|\frac{1}{|\vec{v}|^2}$$

(which also follows from $a_N = \kappa|\vec{v}|^2 = \sqrt{|\vec{a}|^2} = |\vec{a}|$)

Curvature (κ) is a property of the curve & $\therefore$ not necessarily constant. The particle can maintain a constant speed by accelerating normally.

E.g. (not obvious!) $\vec{r}(t) = u\hat{\imath} + \frac{2}{3}u^{3/2}\hat{\jmath}$ where

$u = t^{2/3} - 1$

$$\Rightarrow \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{du}\frac{du}{dt} = \left(\hat{\imath} + u^{1/2}\hat{\jmath}\right)\left(\tfrac{2}{3}t^{-1/3}\right) = \left(\hat{\imath} + [t^{2/3}-1]^{1/2}\hat{\jmath}\right)\left(\tfrac{2}{3}t^{-1/3}\right)$$

$$\Rightarrow \left|\frac{d\vec{r}}{dt}\right| = |\vec{v}| = \sqrt{1^2 + \left([t^{2/3}-1]^{1/2}\right)^2}\,\left|\tfrac{2}{3}t^{-1/3}\right|$$

$$= \tfrac{2}{3}t^{-1/3}\sqrt{1 + t^{2/3} - 1} = \tfrac{2}{3}t^{-1/3}\sqrt{t^{2/3}} = \tfrac{2}{3}t^{-1/3}t^{1/3}$$

$$= 2/3 = \text{constant}$$

But $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[\tfrac{2}{3}t^{-1/3}\left(\hat{\imath} + [t^{2/3}-1]^{1/2}\hat{\jmath}\right)\right]$

$$= -\tfrac{2}{9}t^{-4/3}\,\hat{\imath} + (\text{mess}(t))\,\hat{\jmath}$$

Lots of algebra $\Rightarrow$ $|\vec{a}| = \dfrac{2}{9t\sqrt{t^{2/3}-1}}$ $\neq$ constant.

3. $\vec{r} = \cos t\,\hat{\imath} + 2\sin t\,\hat{\jmath}$ $\Rightarrow \vec{v} = -\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath}$

$\Rightarrow |\vec{v}| = \sqrt{\sin^2 t + 4\cos^2 t} = \sqrt{1+3\cos^2 t}$

$\Rightarrow \vec{a} = -\cos t\,\hat{\imath} - 2\sin t\,\hat{\jmath}$

$\Rightarrow |\vec{a}| = \sqrt{\cos^2 t + 4\sin^2 t} = \sqrt{1+3\sin^2 t}$

a) Since $0 \le \sin^2 t \le 1$ & $0 \le \cos^2 t \le 1$, $\sqrt{1} \le |\vec{v}| \le \sqrt{4}$

& $\sqrt{1} \le |\vec{a}| \le \sqrt{4}$

ie $\boxed{\begin{array}{l} 1 \le |\vec{v}| \le 2 \\ 1 \le |\vec{a}| \le 2 \end{array}}$

b) $\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin t & 2\cos t & 0 \\ -\cos t & -2\sin t & 0 \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + [2\sin^2 t + 2\cos^2 t]\hat{k} = 2\hat{k}$

$\Rightarrow |\vec{v} \times \vec{a}| = |2\hat{k}| = 2$ $\Rightarrow$ $\boxed{\kappa = \frac{2}{|\vec{v}|^3} = \frac{2}{(1+3\cos^2 t)^{3/2}}}$

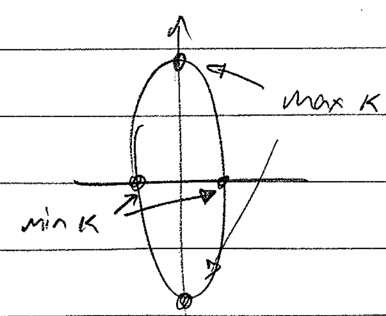
c) Max κ will be when $|\vec{v}|$ is min $\Rightarrow$ $\boxed{\max \kappa = \frac{2}{1} = 2}$

Min κ — — — — — max $\Rightarrow$ $\boxed{\min \kappa = \frac{2}{2^3} = \frac{1}{4}}$

Max κ occurs when $\cos^2 t = 0$ $\Leftrightarrow$ $t = \pi/2, 3\pi/2$ $\Rightarrow \vec{r}(t) = 0\hat{\imath} \pm 2\hat{\jmath}$

Min — — — — $=1$ $\Leftrightarrow$ $t = 0, \pi$ $\Rightarrow \vec{r}(t) = \pm\hat{\imath} + 0\hat{\jmath}$

$\Rightarrow$ particle is at $\boxed{\begin{array}{l} (0, \pm 2) \text{ when } \kappa \text{ is max} \\ (\pm 1, 0) \text{ — — — min} \end{array}}$



4. $\vec{r} = \cos t\,\hat{\imath} + 2t\,\hat{\jmath} + \sin t\,\hat{k}$

$\Rightarrow \vec{v} = -\sin t\,\hat{\imath} + 2\,\hat{\jmath} + \cos t\,\hat{k} \quad \Rightarrow \quad |\vec{v}| = \sqrt{\sin^2 t + 4 + \cos^2 t} = \sqrt{5}$

$\Rightarrow \vec{a} = -\cos t\,\hat{\imath} + 0\,\hat{\jmath} - \sin t\,\hat{k}$

a) $\boxed{\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{-1}{\sqrt{5}}\sin t\,\hat{\imath} + \frac{2}{\sqrt{5}}\,\hat{\jmath} + \frac{1}{\sqrt{5}}\cos t\,\hat{k}}$

$\frac{d\hat{T}}{dt} = \frac{1}{\sqrt{5}}\frac{d\vec{v}}{dt} = \frac{1}{\sqrt{5}}\vec{a} = \frac{-1}{\sqrt{5}}\left[\cos t\,\hat{\imath} + \sin t\,\hat{k}\right]$

$\Rightarrow \left|\frac{d\hat{T}}{dt}\right| = \left|\frac{-1}{\sqrt{5}}\right|\sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{5}}$

$\Rightarrow \boxed{\hat{N} = -\cos t\,\hat{\imath} - \sin t\,\hat{k}}$

$$\hat{B} = \hat{T} \times \hat{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{-1}{\sqrt{5}}\sin t & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\cos t \\ -\cos t & 0 & -\sin t \end{vmatrix} = \frac{-2}{\sqrt{5}}\sin t\,\hat{\imath} - \left[\frac{1}{\sqrt{5}}\sin^2 t + \frac{1}{\sqrt{5}}\cos^2 t\right]\hat{\jmath} + \frac{2}{\sqrt{5}}\cos t\,\hat{k}$$

$\Rightarrow \boxed{\hat{B} = \frac{-1}{\sqrt{5}}\left[2\sin t\,\hat{\imath} + \hat{\jmath} - 2\cos t\,\hat{k}\right]}$

b) $\kappa = \left|\frac{d\hat{T}}{dt}\right|\frac{1}{|\vec{v}|} = \frac{1}{\sqrt{5}}\frac{1}{\sqrt{5}} \Rightarrow \boxed{\kappa = \frac{1}{5}}$

c) $\boxed{\begin{aligned} a_T &= \frac{d}{dt}|\vec{v}| = \frac{d}{dt}\sqrt{5} = 0 \\ a_N &= \kappa|\vec{v}|^2 = \frac{1}{5}(\sqrt{5})^2 = 1 \end{aligned}}$ $\Rightarrow \vec{a} = \hat{N}$ (✓)

5 a) 2

b) 4

c) 5

d) 8