
INSTRUCTIONS: Computers, calculators, books, notes, frying monkeys, *etc.* are not permitted. Some (possibly) useful formulae are attached. Write your name, your instructor's name, and the color of your exam sheet on the front of your bluebook. Work all problems. Start each problem on a **new page**. Show your work clearly and $\boxed{\text{box}}$ your final answer. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

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1. The porosity of the soil in a circular field, $x^2 + y^2 \leq 1$, is given by $p(x, y) = 4x^2 + 2y^2 - 2y$. Find the locations in the field where water will drain the fastest and slowest (*i.e.* the absolute maximum and minimum porosity).

2. A goldsmith wants to make a gold brooch in the shape given in polar coordinates by $r \leq 1 - \sin(\theta)$. The density of gold she is going to use on this brooch design is given by $\delta(x, y) = 2 + 3\sqrt{x^2 + y^2}$. Sketch the shape of the brooch and determine the total amount of gold she will need to use. Hints: 1) the area of this shape is $\frac{3\pi}{2}$; 2) $\sin^2(a) = \frac{1}{2}(1 - \cos(2a))$.

3. A probe is moving through space, being bombarded with cosmic radiation. The strength of the radiation bombardment is given by a function $R(x, y, z)$ and the probe is moving along a path $\mathbf{r}(t)$. At a given time t^* you know that $\mathbf{r}(t^*) = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{v}(t^*) = \mathbf{j} - 3\mathbf{k}$ and $\mathbf{a}(t^*) = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$; you also know that $R(1, -2, -1) = 5$ and $\nabla R|_{(1,-2,-1)} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

   (a) What is the rate of change of the radiation with respect to <u>time</u> that the probe experiences?

   (b) What is the rate of change of the radiation with respect to <u>distance</u> that the probe experiences?

   (c) Suppose the probe is receiving an overdose of radiation, so mission control wants to send a command to change direction. Which direction should they redirect the probe to so that the radiation dosage decreases most rapidly? (You may assume instantaneous communication and redirection.)

   (d) Suppose (ignoring part (c)) that mission control wants to keep the probe at the current level of radiation. Unfortunately, due to budget cuts, the probe was built by students from CSU, rather than CU; consequently, the thrusters that maneuver the probe in the $\mathbf{k}$ direction have broken. Can they still move the probe in a direction that keeps R constant? If not, why not? If so, find the direction.

4. The temperature on the surface of an ellipsoidal object $4x^2 + y^2 + 4z^2 = 16$ is given by $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. Find the maximum and minimum temperatures on the surface of the object. [It may help to know that $\sqrt{3} < 16/9$.]

— **Over** —

5. The density, ρ, of a "particle" of fluid depends on time, t, and position (x, y, z). If the fluid is moving, then the particle's position depends on time $(x, y, z) = (x(t), y(t), z(t))$. Show that the total rate of change of density with respect to time is given by

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\nabla \rho) \cdot \mathbf{v}$$

where $\mathbf{v}$ is the velocity of the fluid.

— Useful and interesting formulae —

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\vec{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\vec{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

$$\hat{\mathbf{T}} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \hat{\mathbf{N}} = \frac{d\hat{\mathbf{T}}/ds}{|d\hat{\mathbf{T}}/ds|} = \frac{d\hat{\mathbf{T}}/dt}{|d\hat{\mathbf{T}}/dt|} \qquad \mathbf{B} = \hat{\mathbf{T}}\times\hat{\mathbf{N}}$$

$$\kappa = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\hat{\mathbf{N}}$$

$$\mathbf{a} = a_T\hat{\mathbf{T}} + a_N\hat{\mathbf{N}} \qquad \text{where} \qquad a_T = \frac{d}{dt}|\mathbf{v}|, \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

$$\frac{df}{ds} = D_{\mathbf{u}}f = (\nabla f)\cdot\mathbf{u}$$

$$\text{Discriminant: } \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$\nabla f = \lambda \nabla g \qquad g = 0$$

$$\begin{aligned} f(x,y) &= f(0,0) + \left(\left.\frac{\partial f}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial f}{\partial y}\right|_{(0,0)} y\right) \\ &+ \frac{1}{2}\left(\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} x^2 + 2\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(0,0)} xy + \left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} y^2\right) \\ &+ \ldots + \frac{1}{n!}\left.\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n\right|_{(0,0)} f + \ldots \end{aligned}$$

$$|E(x,y)| \le \frac{M}{2}\left(|x-x_0| + |y-y_0|\right)^2 \qquad \text{where} \qquad |f_{xx}|, |f_{xy}|, |f_{yy}| \le M$$

$$M = \iint_R \delta(x,y)\,dA \qquad M_x = \iint_R y\delta(x,y)\,dA \qquad M_y = \iint_R x\delta(x,y)\,dA$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

$$I_x = \iint_R y^2\delta(x,y)\,dA \qquad I_y = \iint_R x^2\delta(x,y)\,dA \qquad I_0 = \iint_R (x^2+y^2)\delta(x,y)\,dA$$

$$R_x = \sqrt{I_x/M} \qquad R_y = \sqrt{I_y/M}$$