1. $p = 4x^2 + 2y^2 - 2y$

$\nabla p = 8x\,\hat{i} + (4y-2)\,\hat{j} = \vec{0} \Rightarrow \left.\begin{array}{l} x = 0 \\ y = \frac{1}{2} \end{array}\right\} \rightarrow$ in $x^2+y^2 \le 1 \Rightarrow \boxed{(0, \frac{1}{2})}$

On boundary $x^2 = 1 - y^2 \Rightarrow p = 4(1-y^2) + 2y^2 - 2y$

$= 4 - 2y^2 - 2y \qquad y \in [-1, 1]$

$\Rightarrow p' = -4y - 2 = 0 \Rightarrow y = -\frac{1}{2}$

$\Rightarrow x^2 = 1 - (-\frac{1}{2})^2 = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow x = \pm\frac{\sqrt{3}}{2}$ $\quad \boxed{(\pm\frac{\sqrt{3}}{2}, \frac{-1}{2})}$

Endpoints: $y = \pm 1, x = 0 \Rightarrow p = 2 \mp 2 = \begin{cases} 0 \\ 4 \end{cases}$

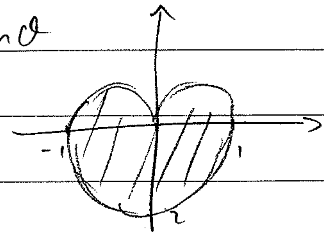
$\Rightarrow$ possibilities:

(x,y)	$p(x,y)$	
$(0,1)$	0	
$(0,-1)$	4	
$(\pm\frac{\sqrt{3}}{2}, \frac{-1}{2})$	$4\frac{1}{2}$	$\leftarrow$ max
$(0, \frac{1}{2})$	$-\frac{1}{2}$	$\leftarrow$ min

$\Rightarrow$ fastest drainage at $(\frac{\pm\sqrt{3}}{2}, \frac{-1}{2})$ $\quad (p = \frac{9}{2})$

slowest " " $(0, \frac{1}{2})$ $\quad (p = \frac{-1}{2})$

2. $r \le 1 - \sin\theta$

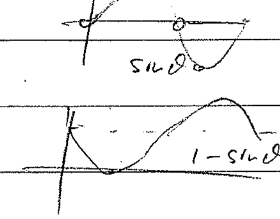


$\theta = 0 \Rightarrow r = 1$

$\theta = \pi/2 \Rightarrow r = 0$

$\theta = \pi \Rightarrow r = 1$

$\theta = 3\pi/2 \Rightarrow r = 2$



$$\text{Total gold} = \iint_R \delta(x,y)\, dA = \iint_R 2 + 3\sqrt{x^2+y^2}\, dA$$

$$= \iint_R 2\, dA + 3\iint_R \sqrt{x^2+y^2}\, dA = 2(\text{Area of } R) + 3\iint_R \sqrt{}\, dA$$

$$= 3\pi + 3\iint_R \sqrt{x^2+y^2}\, dA$$

$$3\iint_R \sqrt{x^2+y^2}\, dA = 3\int_0^{2\pi}\int_0^{1-\sin\theta} r\; r\, dr\, d\theta = \int_0^{2\pi}\int_0^{1-\sin\theta} 3r^2\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[r^3\right]_0^{1-\sin\theta} d\theta = \int_0^{2\pi} (1-\sin\theta)^3\, d\theta$$

$$= \int_0^{2\pi} 1 - 3\sin\theta + 3\sin^2\theta - \sin^3\theta\, d\theta$$

$$= \int_0^{2\pi} 1 - 3\sin\theta + \tfrac{3}{2}(1-\cos(2\theta)) - \sin\theta(1-\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} 5/2 - 4\sin\theta - \tfrac{3}{2}\cos(2\theta) + \sin\theta\cos^2\theta\, d\theta$$

$$= 2\pi\, 5/2 + \overset{0}{\left[\cancel{4\cos\theta}\right]_0^{2\pi}} - \overset{0}{\left[\cancel{\tfrac{3}{4}\sin(2\theta)}\right]_0^{2\pi}} - \overset{0}{\left[\cancel{\tfrac{1}{3}\cos^3\theta}\right]_0^{2\pi}}$$

$$= 5\pi$$

$$\Rightarrow \text{Total gold} = 3\pi + 5\pi = \boxed{8\pi}$$

3 a) $\frac{dR}{dt} = \frac{\partial R}{\partial x}\frac{dx}{dt} + \frac{\partial R}{\partial y}\frac{dy}{dt} + \frac{\partial R}{\partial z}\frac{dz}{dt}$

$= (\nabla R) \cdot \vec{v}$

$= (\hat{\imath} + 3\hat{\jmath} - 2\hat{k}) \cdot (\hat{\jmath} - 3\hat{k}) = 3+6 = \boxed{9}$

b) $\frac{dR}{ds} = \frac{dR}{dt}\frac{dt}{ds} = \frac{dR}{dt}\frac{1}{|\vec{v}|} = (\nabla R) \cdot \frac{\vec{v}}{|\vec{v}|} = (\nabla R) \cdot \hat{T} = D_{\hat{T}} R$

$= 9 \frac{1}{\sqrt{1^2 + (-3)^2}} = \boxed{9/\sqrt{10}}$

c) To decrease R fastest, go in $-\nabla R$ direction

$\Rightarrow \hat{u} = \frac{-1}{\sqrt{1+9+4}}(\hat{\imath} + 3\hat{\jmath} - 2\hat{k}) \Rightarrow \boxed{\hat{u} = \frac{-1}{\sqrt{14}}(\hat{\imath} + 3\hat{\jmath} - 2\hat{k})}$

d) $\vec{u} = a\hat{\imath} + b\hat{\jmath}$ & need $D_{\hat{u}} R = 0 \Rightarrow \nabla R \cdot \hat{u} = 0$

$\Rightarrow \nabla R \cdot \vec{u} = 0$

$\Rightarrow (\hat{\imath} + 3\hat{\jmath} - 2\hat{k}) \cdot (a\hat{\imath} + b\hat{\jmath}) = a + 3b = 0$

$\Rightarrow a = -3b$

$\Rightarrow \vec{u} = -3\hat{\imath} + \hat{\jmath}$ (or some multiple thereof)

$\Rightarrow \boxed{\hat{u} = \frac{\pm 1}{\sqrt{10}}(3\hat{\imath} - \hat{\jmath})}$

4. Optimize $T = 8x^2 + 4yz - 16z + 600$ subject to $4x^2 + y^2 + 4z^2 = 16$

$$\nabla T = 16x\,\hat{i} + 4z\,\hat{j} + (4y-16)\hat{k}$$
$$\nabla g = 8x\,\hat{i} + 2y\,\hat{j} + 8z\,\hat{k}$$

$$\Rightarrow \begin{aligned} 16x &= 8\lambda x \\ 4z &= 2\lambda y \\ 4y-16 &= 8\lambda z \end{aligned} \quad \Rightarrow \boxed{\begin{aligned} 2x &= \lambda x && (1) \\ 2z &= \lambda y && (2) \\ y-4 &= 2\lambda z && (3) \\ 4x^2+y^2+4z^2 &= 16 && (4) \end{aligned}}$$

(1) $\Rightarrow x = 0$ OR $\lambda = 2$

$\underline{x=0}$.

$$\left.\begin{aligned} (2) &\Rightarrow 4z^2 = 2\lambda yz \\ (3) &\Rightarrow y^2 - 4y = 2\lambda yz \end{aligned}\right\} \Rightarrow 4z^2 = y^2 - 4y$$

(4) $\Rightarrow y^2 + 4z^2 = 16$

$\Rightarrow y^2 + y^2 - 4y = 16$

$\Rightarrow 2y^2 - 4y = 16 \Rightarrow y^2 - 2y - 8 = 0$

$\Rightarrow (y-4)(y+2) = 0 \quad \Rightarrow y = 4, -2$

$y = 4 \Rightarrow 4z^2 = 0 \Rightarrow z = 0 \quad \Rightarrow \boxed{(0,4,0)}$

$y = -2 \Rightarrow 4z^2 = 12 \Rightarrow z = \pm\sqrt{3} \quad \Rightarrow \boxed{(0,-2,\pm\sqrt{3})}$

$\underline{\lambda = 2}$ (2) $\Rightarrow 2z = 2y \Rightarrow y = z \overset{(3)}{\Rightarrow} z - 4 = 4z$

$\Rightarrow -4 = 3z \quad \Rightarrow z = y = \frac{-4}{3}$

(4) $\Rightarrow 4x^2 + 5y^2 = 4x^2 + 5\left(\frac{16}{9}\right) = 16 \quad \Rightarrow x^2 = 4 - \frac{5\times 4}{9} = 4 - \frac{20}{9}$

$\Rightarrow x^2 = \frac{36-20}{9} = \frac{16}{9} \quad \Rightarrow x = \pm\frac{4}{3}$

$\Rightarrow \boxed{\left(\frac{\pm 4}{3}, \frac{-4}{3}, \frac{-4}{3}\right)}$

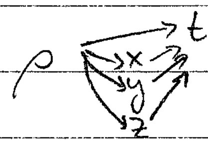
$\Rightarrow$ possibilities:

(x,y,z)	$T(x,y,z)$
$(0,4,0)$	600
$(0,-2,\sqrt{3})$	$600-8\sqrt{3}-16\sqrt{3} = 600-24\sqrt{3}$
$(0,-2,-\sqrt{3})$	$600+8\sqrt{3}+16\sqrt{3} = 600+24\sqrt{3}$
$(\pm\frac{4}{3},-\frac{4}{3},-\frac{4}{3})$	$600+8(\frac{16}{9})+4(\frac{16}{9})+16(\frac{4}{3})$

$$= 600 + \frac{16}{9}(8+4+12) = 600 + 24\left(\frac{16}{9}\right)$$

Since $\frac{16}{9} > \sqrt{3}$, $600 + 24(\frac{16}{9}) > 600 + 24\sqrt{3}$, so

$$\boxed{\begin{array}{l} \text{Max } T \text{ is } 600 + \frac{128}{3} \text{ at } (x,y,z) = (\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}) \\ \text{Min } T \text{ is } 600 - 24\sqrt{3} \text{ at } (x,y,z) = (0,-2,\sqrt{3}) \end{array}}$$

5. 

$$\Rightarrow \frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}\frac{dx}{dt} + \frac{\partial\rho}{\partial y}\frac{dy}{dt} + \frac{\partial\rho}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial\rho}{\partial t} + \left(\frac{\partial\rho}{\partial x}\hat{i} + \frac{\partial\rho}{\partial y}\hat{j} + \frac{\partial\rho}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right)$$

$$= \frac{\partial\rho}{\partial t} + (\nabla\rho)\cdot\frac{d\vec{r}}{dt}$$

$$= \frac{\partial\rho}{\partial t} + (\nabla\rho)\cdot\vec{v}$$

QED.