INSTRUCTIONS: Computers, calculators, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (20 points) Consider the multivariable function $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$. Find the (x,y) coordinates of all the local maxima, local minima, and saddle points for $f(x,y)$. For each point you find, be sure to clearly state which of these three types the point is.

2. (20 points) Determine the extreme values of the multivariable function $f(x,y) = \dfrac{x^2}{4} + \dfrac{y^2}{9}$ on the curve defined by $\dfrac{(x-6)^2}{9} + \dfrac{y^2}{4} = 1$. For each point you find, be sure to state whether the function has a local maxima or minima at that location, and the value of $f(x,y)$.

3. (20 points) Cynthia is paddling in her new canoe along the path $\mathbf{r}(t)$ in Suluclac swamp. The temperature distribution that morning in the swamp is $T(x,y,z)$. At some time t^* (and only at this particular time), you know that $\mathbf{r}(t^*) = 1\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$, $\mathbf{v}(t^*) = 2\,\mathbf{i} + 1\,\mathbf{j} + 2\,\mathbf{k}$, and $\mathbf{a}(t^*) = 3\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$. Furthermore, you know that $\nabla T|_{(1,2,3)} = 2\,\mathbf{i} + 2\,\mathbf{j} + 5\,\mathbf{k}$, and $T(1,2,3) = 10$.

   (a) As Cynthia paddles past location $\mathbf{r}(t^*)$, at what rate is the temperature T changing with respect to time?

   (b) As she paddles past location $\mathbf{r}(t^*)$ at what rate is the temperature T changing with respect to distance?

   (c) If the Cynthia continues on her original path $\mathbf{r}(t)$ for a short interval of time $\Delta t = 0.1$, by approximately how much does the temperature change.

   (d) On the other hand, suppose at time t^* Cynthia suddenly sees her friend Cecile, and starts to paddle towards her in a direction that happens to be the direction of the greatest rate of increase of T. Assuming Cynthia maintains her same speed, by approximately how much does the temperature change after she paddles for $\Delta t = 0.1$.

4. (20 points) It's your first day at your new job and your supervisor comes in with a few questions—just to check out the new employee and see what she is getting in exchange for that huge paycheck.

   (a) She asks you to calculate the *second order* Taylor approximation to the function $f(x,y) = \sin(x+y) + (x+y)$ near the *origin*. What is the second order approximation?

   (b) Then, in keeping with the tradition of supervisors everywhere, she changes her mind and states that now she wants the *first order* approximation to $f(x,y)$ *near the point P located at* $\left(\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$, and an estimate of the associated error. She casually adds that she will only use the approximation for values of x and y such that $\left|x - \dfrac{\pi}{4}\right| \leq 0.1$ and $\left|y - \dfrac{\pi}{4}\right| \leq 0.1$. You confidently scribble away on a piece of scratch paper for a few moments. What linearization and error bound do you give her?

5. (20 points) Jeremy, an Applied Math undergraduate and mountain biking contestant in the last Summer Olympics, spent countless hours working with Kim Kurry, the famed mountain bike mechanic. Kim has developed a monitor that can be mounted on Jeremy's bike to give him information on the maximum possible speed around a corner, V_{max}. He has also developed three different sensors that can measure either the tire pressure, p, the spoke tension, t, or the inner tube radius, r. Unfortunately, due to size and weight restrictions, the monitor can only hold two of the three sensors. But,

this is okay because Jeremy and Kim have determined that there are three ways to calculate the maximum speed based on readings from only two sensors. Specifically:

1) $V_{max} = p\,r^3$ # 2) $V_{max} = r^4/t^2$ # 3) $V_{max} = p^4\,t^6$.

Each of these three calculations has been burned onto a separate microprocessor chip. Jeremy simply has to select one of the chips, the two appropriate sensors, and snap them into the on-board monitor to have real-time information on how fast he can take turns.

Jeremy has arrived at a race one day early to test all the equipment and has found that all three sensors for measuring p, t and r are reading 1% high.

(a) Jeremy, who did quite well in Calculus III, quickly determines which chip (and hence two sensors) he should choose to give him the smallest percentage error for V_{max}. Which chip (#1, #2 or #3) and two sensors (p, t or r) does he select?

(b) Kim frantically text messages Jeremy from Boulder and explains that if Jeremy picks one kind of sensor (p, t or r), then Kim could build a more sensitive model and still have time to ship it overnight to Jeremy. The new sensor would have half the error of the original, so it would only read 0.5% high on the day of the race. Based on your answer in part (a), which kind of sensor should Jeremy pick? What is the resulting percent error in V_{max}? (Be sure to state whether the percent error will be high or low.)

Projections and distances

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + \Big[(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\Big] \\ &\quad + \frac{1}{2!}\Big[(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\Big] \\ &\quad + \frac{1}{3!}\Big[(x-x_0)^3 f_{xxx}(x_0,y_0) + 3(x-x_0)^2(y-y_0)f_{xxy}(x_0,y_0) \\ &\qquad + 3(x-x_0)(y-y_0)^2 f_{xyy}(x_0,y_0) + (y-y_0)^3 f_{yyy}(x_0,y_0)\Big] + \cdots \end{aligned}$$

Linear approximation error

$$|E(x,y)| \le \frac{1}{2}M(|x-x_0| + |y-y_0|)^2, \qquad \text{where } \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \le M$$