Calc. III EXAM 2 Solns FALL 06

(1) $f = 6x^2 - 2x^3 + 3y^2 + 6xy$

To find critical pts! $f_x = 0$ and $f_y = 0$

$f_x = 12x - 6x^2 + 6y = 0$ and $f_y = 6y + 6x = 0$

Since $6x + 6y = 0$, f_x reduces to

$$6x - 6x^2 = 0$$

or $x(1-x) = 0$

If $x = 0$ then from $x + y = 0$ we have $y = 0$

If $x = 1$ then from $x + y = 0$ we have $y = -1$

So crit. pts are $(0,0)$ and $(1,-1)$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = (12 - 12x)(6) - (6)^2$$

$D\big|_{(0,0)} = 12 \cdot 6 - 6 \cdot 6 = 36 \Rightarrow$ local min. and $f(0,0) = 0$

$f_{xx} > 0$

$D\big|_{(1,-1)} = 0 \cdot 6 - 6 \cdot 6 = -36 \Rightarrow$ sad. pt

(2) $f(x,y) = \frac{x^2}{4} + \frac{y^2}{9}$ and $g(x,y) = \frac{(x-6)^2}{9} + \frac{y^2}{4} - 1 = 0$

$$\nabla f = \lambda \nabla g \Rightarrow \left(\frac{x}{2}\hat{i} + \frac{2y}{9}\hat{j}\right) = \lambda\left(\frac{2(x-6)}{9}\hat{i} + \frac{y}{2}\hat{j}\right)$$

So

(1) $\frac{x}{2} = \lambda \frac{2}{9}(x-6)$ (2) $\frac{2}{9}y = \lambda \frac{y}{2}$ (3) is $g=0$

From (2) $y\left(\frac{4}{9} - \lambda\right) = 0$ leads to two cases

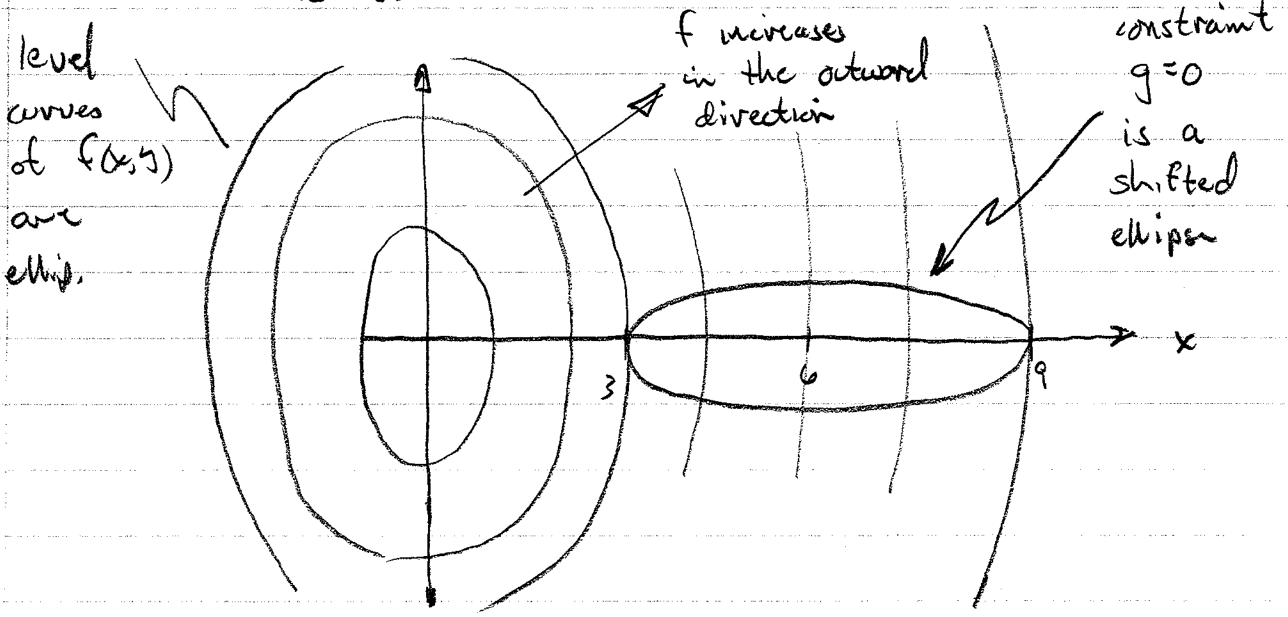
Case (A) $\lambda = \frac{4}{9}$ Substituting $\lambda = 4/9$ into (1) and solving for x gives $x = -32/15$ which is not on the constraint curve $g=0$. Hence we elliminate this case.

Case (B) $y=0$ Substituting $y=0$ into (3) yields $x = 3, 9$.

- If $x=3$, then (1) $\Rightarrow \lambda = -9/4$
- If $x=9$, then (1) $\Rightarrow \lambda = 24/4$

Value of $f\big|_{(3,0)} = 9/4$ and $f\big|_{(9,0)} = 81/4$

Should make sense because



$$\text{③}\quad \frac{dT}{dt} = \frac{d}{dt} \underbrace{T\big(x(t), y(t), z(t)\big)}_{\text{T while on path } \underline{r}(t)} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

a)
$$= \nabla T \circ \frac{d\underline{r}}{dt} = \nabla T \circ \underline{V}$$

or
$$\frac{dT}{dt} = \frac{dT}{ds}\cdot\frac{ds}{dt} = \left(\nabla T \circ \hat{U}\right)|\underline{V}| \quad \text{where } \hat{U} = \frac{\underline{V}}{|\underline{V}|} \text{ is direction of motion}$$
$$= \nabla T \circ \underline{V}$$

$$\left.\frac{dT}{dt}\right|_{t^*} = \left.\nabla T\right|_{t^*} \circ \left.\underline{V}\right|_{t^*} = \left(2\hat{i} + 2\hat{j} + 5\hat{k}\right)\circ\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$$
$$= 4 + 2 + 10 = 16 \ \left({}^\circ/\text{unit time}\right)$$

b)
$$\frac{dT}{ds} = \frac{dT}{dt}\cdot\frac{dt}{ds} = \frac{dT}{dt}\cdot\frac{1}{|\underline{V}|} = 16\left(\frac{\circ}{\text{time}}\right)\cdot\frac{1}{\sqrt{2^2+1^2+2^2}}\left(\frac{\text{time}}{\text{unit length}}\right)$$
$$= \frac{16}{3}\ \left({}^\circ/\text{unit length}\right)$$

c)
$$\Delta T \approx \left(\frac{dT}{dt}\right)\Delta t = 16\left(\frac{\circ}{\text{unit time}}\right)\cdot 0.1\,(\text{time unit}) = 1.6^\circ$$

d) Moving in direction of ∇T means $\hat{U} = \frac{\nabla T}{|\nabla T|} = \frac{2\hat{i} + 2\hat{j} + 5\hat{k}}{\sqrt{2^2 + 2^2 + 5^2}}$

$$= \frac{1}{\sqrt{33}}\left(2\hat{i} + 2\hat{j} + 5\hat{k}\right)$$

$$\Delta T \approx \left(\frac{dT}{dt}\right)\Delta t = \frac{dT}{ds}\cdot\frac{ds}{dt}\Delta t = \left(\nabla T \circ \hat{U}\right)|\underline{V}|\,\Delta t$$

$$= \left(\nabla T \circ \frac{\nabla T}{|\nabla T|}\right)|\underline{V}|\,\Delta t = |\nabla T|\cdot|\underline{V}|\,\Delta t$$

$$= \frac{1}{\sqrt{33}}\cdot\frac{1}{3}\,0.1 = \frac{0.1}{3\sqrt{33}}\ {}^\circ$$

(4)
a)

$$f(x,y) \approx f(x_0,y_0) + (x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0) + \frac{1}{2}\left[(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\right]$$

$$f(0,0) = \left[\sin(x+y) + (x+y)\right|_{(0,0)} = 0$$

$$f_x(0,0) = \left(\cos(x+y) + 1\right|_{(0,0)} = 1+1 = 2$$

$$f_y(0,0) = \left(\cos(x+y) + 1\right|_{(0,0)} = 1+1 = 2$$

$$f_{xx}^{(0,0)} = -\sin(x+y)\Big|_{(0,0)} = 0$$

$$f_{yy}^{(0,0)} = -\sin(x+y)\Big|_{(0,0)} = 0 \qquad f_{xy} = -\sin(x+y)\Big|_{(0,0)} = 0$$

so near the origin

$$f(x,y) = \sin(x+y) + (x+y) \approx$$

$$0 + (x-0)2 + (y-0)2 + \frac{1}{2}\left[(x-0)^2\cdot 0 + 2(x-0)(y-0)\cdot 0 + (y-0)^2\cdot 0\right]$$

$$= 2x + 2y \leftarrow$$

4) b) near $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ the 1^{st} order approx. is

$$f(x,y) \approx f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right) f_x\left(\frac{\pi}{4}, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right) f_y\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$$

$$f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right) + \left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \left(1 + \frac{\pi}{2}\right)$$

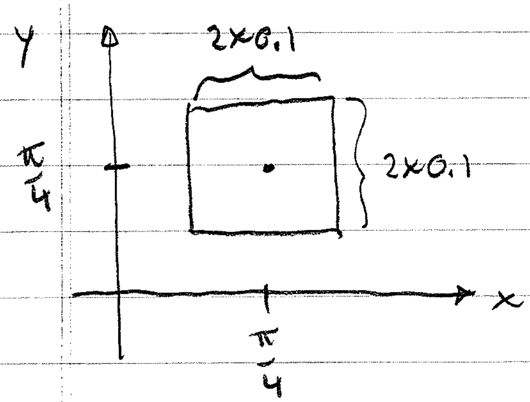
$$f_x\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4} + \frac{\pi}{4}\right) + 1 = 1$$

$$f_y\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}, \frac{\pi}{4}\right) + 1 = 1$$

So near $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

$$\sin(x+y) + (x+y) \approx \left(1 + \frac{\pi}{2}\right) + \left(x - \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right) \leftarrow$$

Error est.



$$\left.\begin{aligned} f_{xx} &= -\sin(x+y) \\ f_{yy} &= -\sin(x+y) \\ f_{xy} &= -\sin(x+y) \end{aligned}\right\}$$ Pick $M = 1$ as max mag. of these

$$|\text{error}| \le \frac{M}{2}\left(\left|x - \frac{\pi}{4}\right| + \left|y - \frac{\pi}{4}\right|\right)^2 = \frac{1}{2}\left(0.1 + 0.1\right)^2$$

$$= \frac{0.04}{2} = 0.02 \leftarrow$$

(5) a) Chip #1 $V = pr^3$ so $\Delta V = \frac{\partial V}{\partial p}\Delta p + \frac{\partial V}{\partial r}\Delta r$

$$= r^3 \Delta p + p 3 r^2 \Delta r$$

so

$$\frac{\Delta V}{V} = \frac{r^3 \Delta p + 3 p r^2 \Delta r}{p r^3} = \frac{\Delta p}{p} + 3\frac{\Delta r}{r}$$

Chip #2 $V = r^4/t^2$ so $\Delta V = \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial t}\Delta t$

$$= \frac{4r^3}{t^2}\Delta r - \frac{2r^4}{t^3}\Delta t$$

and so

$$\frac{\Delta V}{V} = \frac{\frac{4r^3}{t^2}\Delta r - \frac{2r^4}{t^3}\Delta t}{(r^4/t^2)} = 4\frac{\Delta r}{r} - 2\frac{\Delta t}{t}$$

Chip #3 $V = p^4 t^6$ will get $\frac{\Delta V}{V} = 4\frac{\Delta p}{p} + 6\frac{\Delta t}{t}$

Since each sensor reads 1% high $\frac{\Delta p}{p} = \frac{\Delta t}{t} = \frac{\Delta r}{r} = 0.01$

Then for chip #1 $\frac{\Delta V}{V} = 0.01 + 3(0.01) = 0.04 \Rightarrow$ 4% high

chip #2 $\frac{\Delta V}{V} = 4(0.01) - 2(0.01) = 0.02 \Rightarrow$ 2% high

chip #3 $\frac{\Delta V}{V} = 4(0.01) + 6(0.01) = 0.10 \Rightarrow$ 10% high

$\Rightarrow$ Pick chip #2 $\leftarrow$

b) Chip #2 is most sensitive to relative changes in r, so select new sensor for r with rel error $\frac{\Delta r}{r} = 0.005$

Then $\frac{\Delta V}{V} = 4(0.005) - 2(0.01) = 0$ (no error!)