INSTRUCTIONS: Computers, calculators, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (25 points) Find the center of mass of the thin plate with density (mass per unit area) $\delta(x, y) = 1 + y$, bounded by the curves $x = 2y - y^2$, and $x = y^2$.

2. (25 points) Consider the integral

$$I = \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \int_{\phi=0}^{\frac{\pi}{2}} \rho^2 \sin\phi \; d\phi \; d\rho \; d\theta + \int_{\theta=0}^{2\pi} \int_{\rho=1}^{\sqrt{2}} \int_{\phi=\cos^{-1}\left(\frac{1}{\rho}\right)}^{\frac{\pi}{2}} \rho^2 \sin\phi \; d\phi \; d\rho \; d\theta \, .$$

   (a) Make a clear sketch of the region of integration in the xyz-coordinate system. (If you have trouble with this, you may "buy" a sketch of the region of integration for 5 points.)

   (b) Express I in spherical coordinates using the order $d\rho \, d\phi \, d\theta$.

   (c) Express I in cylindrical coordinates using the order $dr \, dz \, d\theta$.

   (d) Express I in cylindrical coordinates using the order $dz \, dr \, d\theta$.

   (e) Evaluate one of the integrals above to determine the value of I.

3. (25 points) Consider the integral

$$I = \iint_{R_{xy}} \sqrt{4x - 2y} \, (-2x + 2y) \; dx \, dy$$

   where R_{xy} is the triangular region in the xy-plane bounded by the lines $y = 2x$, $y = x + 1$, and $y = \dfrac{3}{2}x$.

   (a) The substitution $u = 4x - 2y$, $v = -2x + 2y$, makes this integral computable. Find x and y in terms of u and v under this substitution. Be sure to check this because the rest of the problem depends on this result!

   (b) In order to compute the integral correctly, we need to transform the original region R_{xy} into its corresponding region R_{uv} in the uv-plane. Make two clear sketches, one of the original region of integration R_{xy} in the xy-plane, and one of the new region of integration R_{uv} in the uv-plane. Be sure to label all axes, boundaries, intersection points, etc.

   (c) Rewrite the integral I over the region R_{uv} in the uv-plane, in terms of u and v.

   (d) Evaluate I in terms of u and v.

4. (25 points) Brendan, your Calculus III tutor and Applied Math Mrs. Pac-Man legend, is completely obsessed with the region R, given by all points (x, y) not in the first quadrant, such that $x^2 + y^2 \leq 1$. To help you practice for the next exam, Brendan has prepared some exercises for you. Specifically, he wants you to do some line integrals around the counter clock-wise path C bounding the region R. **Be sure to look at the pictures of R and C on the next page.** Being a huge Mrs. Pac-Man fan, you agree to do anything Brendan asks of you.

   (a) Brendan wants you to find the total mass of a thin wire along the contour C with density (mass per unit length) $\delta(x, y) = 1 + x\,e^y$.

   (b) Now Brendan wants you to find the work (circulation) along the path C for the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$.

   (c) Finally, Brendan wants you to find the flux across C, using the same vector field as part (b).

OVER

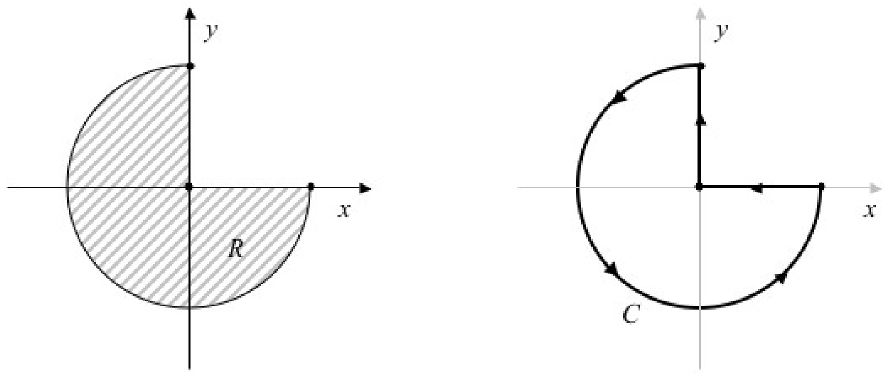

Figure 1: The pac-man shaped region R and corresponding clockwise oriented contour C.

Projections and distances $\quad \text{proj}_{\mathbf{A}}\mathbf{B} = \left(\dfrac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \dfrac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\dfrac{\mathbf{n}}{|\mathbf{n}|}\right|$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Polar coordinates $\quad x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2+y^2 \qquad dA = dx\,dy = r\,dr\,d\theta$

Cylindrical and spherical coordinates

Cylindrical to Rectangular	Spherical to Cylindrical	Spherical to Rectangular
$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$r = \rho\sin\phi$ $z = \rho\cos\phi$ $\theta = \theta$	$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$

$$dV = dx\,dy\,dz = dz\,r\,d\theta\,dr = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Substitutions in multiple integrals

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\left(x(u,v),y(u,v)\right)\,|J(u,v)|\,du\,dv \quad \text{where} \quad J(u,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Mass, moments, and center of mass $\quad$ Mass $\quad M = \iint_R \delta\,dA$

Moments $\quad M_x = \iint_R y\,\delta\,dA \qquad M_y = \iint_R x\,\delta\,dA \qquad$ Center of mass $\quad \bar{x} = M_y/M \qquad \bar{y} = M_x/M$

Flow and flux $\quad$ Flux $= \int_C \mathbf{F}\cdot\mathbf{n}\,ds \qquad$ Flow $= \int_C \mathbf{F}\cdot\mathbf{T}\,ds$