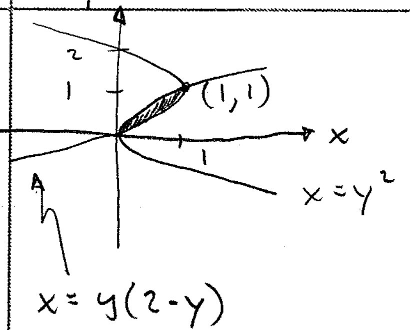
(1)



Density $\delta(x,y) = 1+y$

total mass $M = \iint_R \delta\, dA = \int_{y=0}^{1} \int_{x=y^2}^{y(2-y)} (1+y)\, dx\, dy$

$$= \int_{y=0}^{1} (1+y)\left[(2y-y^2) - y^2\right] dy = \int_{y=0}^{1} 2(y-y^3)\, dy$$

$$= 2\left(\frac{y^2}{2} - \frac{y^4}{4}\right)\Big|_0^1 = \frac{1}{2}$$

$$M_x = \iint_R y\,\delta\, dA = \int_{y=0}^{1}\int_{x=y^2}^{(2-y)y} (y+y^2)\, dx\, dy = \int_{y=0}^{1} (y+y^2)(2y-2y^2)\, dy$$

$$= \int_{y=0}^{1} (2y^2 - 2y^4)\, dy = 4/15$$

$$M_y = \iint x\,\delta\, dA = \int_{y=0}^{1} \int_{x=y^2}^{y(2-y)} (1+y)\, x\, dx\, dy = \cdots = \int_{y=0}^{1} 2(y^2-y^4)\, dy$$

$$= 2\left(\frac{1}{3} - \frac{1}{5}\right) = 2\cdot\frac{2}{15} = 4/15$$

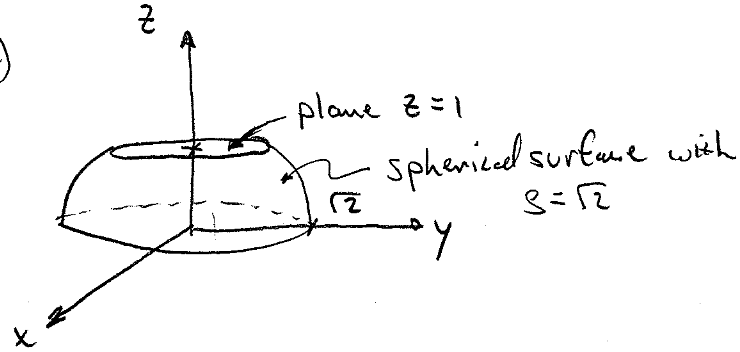
Finally $\underbrace{M_x = \iint_R y\,\delta\, dA}_{4/15} = \bar{y} \underbrace{\iint_R \delta\, dA}_{1/2} \Rightarrow \bar{y} = \frac{4}{15}\cdot 2 = \frac{8}{15}$ ←

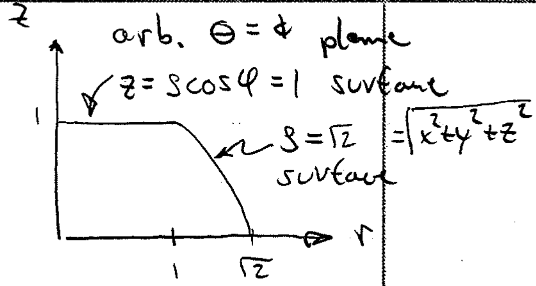
and

$$M_y = \underbrace{\iint_R x\,\delta\, dA}_{4/15} = \bar{x} \underbrace{\iint_R \delta\, dA}_{1/2} \Rightarrow \bar{x} = \frac{4}{15}\cdot 2 = 8/15 \leftarrow$$

(2)

a)





b)

$$I = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi/4}\int_{\rho=0}^{\sec\phi} \rho^2 \sin^2\varphi \, d\rho \, d\varphi \, d\theta + \int_{\theta=0}^{2\pi}\int_{\rho=\pi/4}^{\pi/2}\int_{\rho=0}^{\sqrt{2}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

c)

$$I = \int_{\theta=0}^{2\pi}\int_{z=0}^{1}\int_{r=0}^{\sqrt{2-z^2}} r \, dr \, dz \, d\theta = \int_{\theta=0}^{2\pi}\int_{z=0}^{1} \frac{1}{2}(2-z^2) \, dz \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{5}{6} \, d\theta = \frac{5}{6}\cdot 2\pi = \frac{5}{3}\pi \longleftarrow$$

d)

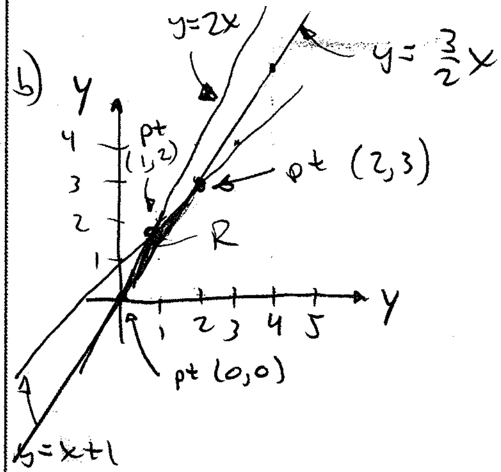
$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{1} r \, dz \, dr \, d\theta + \int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{2}}\int_{z=0}^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta$$

e) See part (c). $I = \frac{5}{3}\pi$

(3)

a) $u = 4x - 2y$, $v = -2x + 2y$ Adding then $\Rightarrow$ $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u+2v)$

b)



To convert boundaries:

(1) $y = 2x$

$\frac{1}{2}(u+2v) = (u+v)$

$u + 2v = 2u + 2v$

$u = 0$

(2) $y = \frac{3}{2}x$

$\frac{1}{2}(u+2v) = \frac{3}{2} \cdot \frac{1}{2}(u+v)$

$2u + 4v = 3u + 3v$

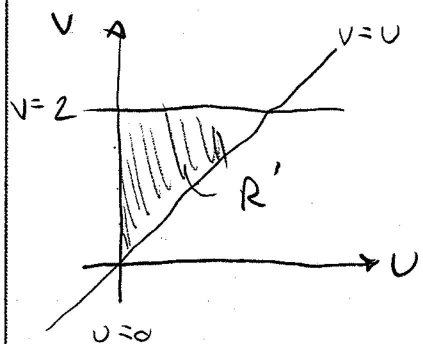
$v = u$

(3) $y = x + 1$

$\frac{1}{2}(u+2v) = \frac{1}{2}(u+v) + 1$

$u + 2v = u + v + 2$

$v = 2$



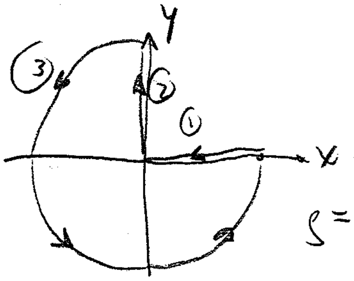
c+d) $$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$I = \iint_{R_{xy}} \sqrt{4x-2y}\,(-2x+2y)\,dx\,dy = \iint_{R_{uv}} \sqrt{u} \cdot v \, |J(u,v)| \, du\,dv$$

$$= \int_{v=0}^{2} \int_{u=0}^{v} \sqrt{u} \cdot v \, \frac{1}{4} \, du\,dv = \frac{1}{4} \int_{v=0}^{2} v \cdot \frac{2}{3} v^{3/2} dv = \frac{1}{6} \cdot \frac{2}{7} v^{7/2} \Big|_0^2$$

$$= \frac{1}{21} 2^{7/2} = \frac{8}{21}\sqrt{2}$$

(4)



path (1) $\underline{r}_1 = (1-t)\hat{i}$ $\quad 0 \le t \le 1$

$\underline{r}_2 = t\,\hat{j}$ $\quad 0 \le t \le 1$

$\rho = 1 + x e^y$

$\underline{r}_3 = \cos t\,\hat{i} + \cos t\,\hat{j}$ $\quad \frac{\pi}{2} \le t \le 2\pi$

$|\underline{v}| = 1$ on path 3

$\underline{F} = x\hat{i} + y\hat{j} + 0\hat{k}$

$\hat{T}_{\text{on } C_3} = -\sin t\,\hat{i} + \cos t\,\hat{j}$

a) $$M_{tot} = \int_{C_1+C_2+C_3} \rho\, ds$$

$$= \underbrace{\int_{t=0}^{1} \left[1 + (1-t)e^0\right] \cdot 1 \cdot dt}_{C_1} + \underbrace{\int_{t=0}^{1} (1 + 0e^t) \cdot 1 \cdot dt}_{C_2}$$

$$+ \underbrace{\int_{t=\pi/2}^{2\pi} (1 + \cos t \cdot e^{\sin t}) \cdot 1 \cdot dt}_{C_3}$$

$$= \left(2t - \frac{t^2}{2}\right|_0^1 + \left(t\right|_0^1 + \left(t + e^{\sin t}\right|_{t=\pi/2}^{t=2\pi}$$

$$= \frac{3}{2} + 1 + \left(\frac{3\pi}{2} + 1 - e\right) = \frac{7}{2} - e + \frac{3\pi}{2}$$

b) $$\oint_{C_1+C_2+C_3} \underline{F} \circ \hat{T}\, ds = \underbrace{\int_{t=0}^{1} \left[(1-t)\hat{i} + 0\hat{j}\right] \circ (-\hat{i})\, dt}_{C_1} + \underbrace{\int_{t=0}^{1} (0\hat{i} + t\hat{j}) \circ (\hat{j})\, dt}_{C_2}$$

$$+ \underbrace{\int_{t=\pi/2}^{2\pi} (\cos t\,\hat{i} + \sin t\,\hat{j}) \circ (-\sin t\,\hat{i} + \cos t\,\hat{j})\, dt}_{C_3}$$

$$= \underbrace{\int_{t=0}^{1} (t-1)\, dt}_{C_1} + \underbrace{\int_{t=0}^{1} t\, dt}_{C_2} + \int_{t=\pi/2}^{2\pi} 0\, dt = \dots = -\frac{1}{2} + \frac{1}{2} = 0$$

Summary:

Flow on $C_1 = -\frac{1}{2}$

" " $C_2 = \frac{1}{2}$

" " $C_3 = 0$

Net flow $= 0$

c) $$\oint_{C_1+C_2+C_3} \underline{F}\circ \hat{n}\, ds = \int_{t=0}^{1} \underbrace{\left[(1-t)\hat{\imath}+0\hat{\jmath}\right]\circ \hat{\jmath}}_{=0} dt + \int_{t=0}^{1} \underbrace{\left[0\hat{\imath}+t\hat{\jmath}\right]\circ \hat{\imath}}_{=0} dt$$

$$+ \int_{t=\pi/2}^{2\pi} \left[\cos t\,\hat{\imath} + \sin t\,\hat{\jmath}\right]\circ \underbrace{\left[\cos t\,\hat{\imath} + \sin t\,\hat{\jmath}\right]}_{\hat{n} \text{ to } C_3} dt$$

$$= \int_{t=\pi/2}^{2\pi} dt = 2\pi - \frac{\pi}{2} = \frac{3}{2}\pi \longleftarrow$$