

INSTRUCTIONS: Computers, calculators, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (30 Points) Consider the multi-variable functions $f(x, y) = \frac{x}{x+y}$ and $g(x, y) = \ln(x+y) - \ln(y-x)$.
 - (a) Sketch the domain and state the range of $f(x, y)$ and $g(x, y)$.
 - (b) Clearly state whether each domain boundary is open or closed.
 - (c) For each of the limits $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ and $\lim_{(x,y) \rightarrow (1,1)} g(x, y)$, find the limit L , or state that it "Does not exist." Show work to support your claims.

2. (30 Points) Maribeth has designed and built a brand new jetpack and is showing it off at the Idaho State Fair. With the jetpack strapped to her back, she flies through the air along the path $\mathbf{r}(t) = \sqrt{8}t \mathbf{i} + (1 - \cos(t)) \mathbf{j} + \sin(t) \mathbf{k}$ for time t from 0 to π seconds. The efficiency, E , of the jetpack can be described as a function of oxygen concentration, c , and altitude, z , by the function $E(c, z) = 2z - c$. The oxygen concentration in the air around the fair grounds, as a function of location, is the unspecified function $c(x, y, z)$.
 - (a) Calculate the rate of change of efficiency E , with respect to time, that Maribeth's jetpack experiences. You may leave your answer in terms of partial derivatives of $c(x, y, z)$ with respect to x , y or z .
 - (b) Calculate the rate of change of efficiency E , with respect to distance, that Maribeth's jetpack experiences. You may leave your answer in terms of partial derivatives of $c(x, y, z)$ with respect to x , y or z .

3. (30 Points) Captain Stubing and crew of "The Love Boat" are making a reunion cruise along the circular path $x^2 + y^2 = 1$, where x stands for miles to the east and y stands for miles to the north. "The Love Boat" will be moving at a uniform speed of 20 miles per hour. You are the new crew member with the title "Aid to the Captain on all Matters Mathematical." You make a quick check of the website *OceanTempDistributions.com* and find that the current temperature of the sea in $^{\circ}\text{F}$, as a function of location, is $T(x, y) = 72 + y^2 - x$.
 - (a) The cruise line company has a promised its clients ocean temperatures between 71°F and 73°F during the entire cruise. Find the minimum and maximum ocean temperatures "The Love Boat" will experience during this voyage, as well as the x - y coordinates where these maximum and minimum temperatures occur. Report to Captain Stubing whether the path he is taking is TOO HOT, TOO COLD, or JUST RIGHT.
 - (b) As you pass over the location $(0, 1)$, at what rate is the temperature changing in $^{\circ}\text{F}/\text{mile}$?
 - (c) How fast is the temperature changing in $^{\circ}\text{F}/\text{hour}$?

4. (30 Points) Compute the circulation of the vector field $\mathbf{F} = z \mathbf{i} + e^{(y^2)} \mathbf{j} + x \mathbf{k}$ over the closed path C composed of sections C_1 , C_2 , C_3 , and C_4 defined by

$$\begin{aligned}
 C_1 & : \{ \mathbf{r}_1(t) = t \mathbf{i} + t \mathbf{k}, & 0 \leq t \leq 1 \} \\
 C_2 & : \{ \mathbf{r}_2(t) = \mathbf{i} + t \mathbf{j} + \mathbf{k}, & 0 \leq t \leq 1 \} \\
 C_3 & : \{ \mathbf{r}_3(t) = (1-t) \mathbf{i} + \mathbf{j} + (1-t) \mathbf{k}, & 0 \leq t \leq 1 \} \\
 C_4 & : \{ \mathbf{r}_4(t) = (1-t) \mathbf{j}, & 0 \leq t \leq 1 \}.
 \end{aligned}$$

Note that for any one of the paths, the \mathbf{i} component is equal to the \mathbf{k} component which means that the closed loop is actually on the surface $z = x$.

5. (20 points) Consider the object bounded below by the surface $\rho = 1$, for $z \geq 0$, and above by the surface $\rho = 1 + \cos(\phi)$.
- Make a clear sketch of the cross-section of the object in the y - z plane.
 - Calculate the volume of the object using the order $d\rho d\phi d\theta$.
 - Set up, but do not evaluate, the integral to calculate the volume of the object using the order $d\theta d\phi d\rho$.

6. (30 Points) Consider the integral

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dx dy$$

where R is the region in the first quadrant of the xy -plane bounded by the hyperbolas $xy = 1$, $xy = 4$ and the lines $y = x$, $y = 9x$. You can use the transformation $x = u/v$ and $y = uv$ to rewrite the integral.

- Sketch the original region of integration, R , in the xy -plane. Clearly label the boundaries.
 - Sketch the new region of integration, G , in the uv -plane and clearly label the boundaries.
 - Finally, evaluate the integral in terms of u and v over G .
7. (30 Points) Consider an object bounded on top by the surface $x^2 + y^2 + z^2 = R^2$ and the bottom by the surface $z = \frac{1}{R}(x^2 + y^2) - R$.
- Find the parameterization of the curve C defined by the intersection of the two surfaces.
 - Calculate the flux of the field $\mathbf{F} = (1 + z)\mathbf{i} + z\mathbf{j} + yz\mathbf{k}$ around the path C .
 - Calculate the circulation of \mathbf{F} around the same path.
 - If one (or both) of the results above can be verified using any theorem(s) from Calculus III, state the theorem(s) and set up (but do not evaluate) the appropriate alternate calculation(s). Otherwise, clearly write "Result cannot be confirmed."

OVER

Projections and distances $\text{proj}_{\mathbf{A}} \mathbf{B} = \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \right) \mathbf{A}$ $d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$ $d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}| dt \quad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \quad \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N} \quad \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|1 + (f'(x))^2|^{3/2}} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{|\dot{x}^2 + \dot{y}^2|^{3/2}} \quad \tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

$$\mathbf{a} = a_N \mathbf{N} + a_T \mathbf{T} \quad a_T = \frac{d|\mathbf{v}|}{dt} \quad a_N = \kappa |\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

The Second Derivative Test

Suppose $f(x, y)$ and its first and second partial derivatives are continuous in a disk centered at (a, b) and $f_x(a, b) = f_y(a, b) = 0$. Let $D = f_{xx}f_{yy} - f_{xy}^2$.

1. If $D > 0$ and $f_{xx} < 0$ at (a, b) , then f has a local maximum at (a, b) .
2. If $D > 0$ and $f_{xx} > 0$ at (a, b) , then f has a local minimum at (a, b) .
3. If $D < 0$ at (a, b) , then f has a saddle point at (a, b) .
4. If $D = 0$ at (a, b) , then the test is inconclusive.

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f) \cdot \mathbf{u} \quad f_{xx}f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda \nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$f(x, y) = f(x_0, y_0) + \left[(x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) \right]$$

$$+ \frac{1}{2!} \left[(x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0)f_{xy}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0) \right]$$

$$+ \frac{1}{3!} \left[(x - x_0)^3 f_{xxx}(x_0, y_0) + 3(x - x_0)^2(y - y_0)f_{xxy}(x_0, y_0) \right.$$

$$\left. + 3(x - x_0)(y - y_0)^2 f_{xyy}(x_0, y_0) + (y - y_0)^3 f_{yyy}(x_0, y_0) \right] + \dots$$

Linear approximation error

$$|E(x, y)| \leq \frac{1}{2} M (|x - x_0| + |y - y_0|)^2, \quad \text{where } \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \leq M$$

Polar coordinates $x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad dA = dx dy = r dr d\theta$

Cylindrical and spherical coordinates

Cylindrical to Rectangular	Spherical to Cylindrical	Spherical to Rectangular
$x = r \cos \theta$	$r = \rho \sin \phi$	$x = \rho \sin \phi \cos \theta$
$y = r \sin \theta$	$z = \rho \cos \phi$	$y = \rho \sin \phi \sin \theta$
$z = z$	$\theta = \theta$	$z = \rho \cos \phi$

$$dV = dx dy dz = dz r d\theta dr = \rho^2 \sin \phi d\rho d\phi d\theta$$

Substitutions in multiple integrals

$$\iint_R f(x, y) dx dy = \iint_G f(x(u, v), y(u, v)) |J(u, v)| du dv \quad \text{where} \quad J(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

Mass, moments, and center of mass Mass $M = \iint_R \delta dA$

Moments $M_x = \iint_R y \delta dA \quad M_y = \iint_R x \delta dA$ Center of mass $\bar{x} = M_y/M \quad \bar{y} = M_x/M$

Green's Theorem in a plane (The curve C is traversed counterclockwise.)

$$\text{Outward Flux} = \oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_R \nabla \cdot \mathbf{F} dA$$

$$\text{Circulation} = \oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} dA$$

Surface area of level surface $f(x, y, z) = c$ $S = \iint_S d\sigma = \iint_S \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA$

Stoke's Theorem $\oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$

Divergence Theorem of Gauss $\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F} dV$