INSTRUCTIONS: Computers, calculators, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (25 points) Consider the multivariable function $f(x,y) = x^4+y^4-4xy+1$. Find the (x,y) coordinates of all the local maxima, local minima, and saddle points for $f(x,y)$. For each point you find, be sure to clearly state which of these three types the point is.
   A friendly reminder: $a^2 - b^2 = (a-b)(a+b)$.

2. (25 points) Consider the curve C in the xy-plane defined by $y = e^x$.

   (a) Using an appropriate Calculus III technique, determine the coordinates of any points on C such that the **product** of the coordinates, xy, is as small as possible. In addition to your calculations, provide a clear sketch illustrating the problem. Be sure to clearly label any relevant curves and points of interest.

   (b) Using an appropriate Calculus III technique, determine the coordinates of any points on C such that the **sum** of the coordinates, $x + y$, is as small as possible. Again, provide a clear sketch illustrating the problem. Be sure to clearly label any relevant curves and points of interest.

3. (25 points) Nainotwen Moths are noted for their ability to detect very small variations in atmospheric pressure. Elsa, a Nainotwen Moth, is flying along the path $\mathbf{r}(t)$ and the pressure distribution (in mbar) that day is $P(x,y)$. At some time t^* (and only at this particular time), you know that $\mathbf{r}(t^*) = 1\,\mathbf{i} + 2\,\mathbf{j}$, $\mathbf{v}(t^*) = 2\,\mathbf{i} + 2\,\mathbf{j}$, and $\mathbf{a}(t^*) = 2\,\mathbf{i} + 0\,\mathbf{j}$, where the components of $\mathbf{r}$ are measured in meters, the components of $\mathbf{v}$ are measured in meters per second, and the components of $\mathbf{a}$ are measured in meters per second squared. Furthermore, you know that $\nabla P|_{(1,2)} = -2\,\mathbf{i} - 4\,\mathbf{j}$, and $P(1,2) = 42$ mbar.

   (a) As Elsa flies past location $\mathbf{r}(t^*)$, at what rate is the pressure P changing with respect to distance? (Your answer should be in mbar/meter.)

   (b) As Elsa flies past location $\mathbf{r}(t^*)$, at what rate is the pressure P changing with respect to time? (Your answer should be in mbar/second.)

   (c) Later in the day, Elsa realizes that she has passed over another point in space, Q, several times. When she passed over point Q in the $\mathbf{i}$ direction, she noted that the rate of change in pressure with respect to distance was -4 mbar/meter. But when she flew over Q in the $\mathbf{j}$ direction, the rate of change in pressure with respect to distance was -3 mbar/meter. Elsa now flies over Q for a third time in the direction of greatest pressure increase. As she does so, she notices that the rate of change in presure with respect to time is 10 mbar/sec. How fast was she flying on her third trip over Q?

4. (25 points) A person's Body Mass Index is $I = m/h^2$, where m is the body weight in kilograms and h is the body height in meters. Thus, the Body Mass Index for a child of mass $m = 17$ kg and height $h = 1$ m, is $I = 17/1^2 = 17$.

   (a) Estimate (using Calculus III principles) the new Body Mass Index, I, if the child grows to $m = 18$ kg and $h = 1.01$ m.

   (b) Now, (using Calculus III principles) give an upper bound on the error associated with your estimation in part (a).

Projections and distances

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$\begin{aligned} f(x,y) \;=\; & f(x_0,y_0) + \Big[\,(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\Big] \\ & + \frac{1}{2!}\Big[\,(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\Big] \\ & + \frac{1}{3!}\Big[\,(x-x_0)^3 f_{xxx}(x_0,y_0) + 3(x-x_0)^2(y-y_0)f_{xxy}(x_0,y_0) \\ & \qquad + 3(x-x_0)(y-y_0)^2 f_{xyy}(x_0,y_0) + (y-y_0)^3 f_{yyy}(x_0,y_0)\Big] + \cdots \end{aligned}$$

Linear approximation error

$$|E(x,y)| \le \frac{1}{2}M(|x-x_0| + |y-y_0|)^2, \qquad \text{where } \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \le M$$