INSTRUCTIONS: Computers, calculators, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

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1. (25 points) Cynthia, a Calculus III student, is trying to determine the moment about the x-axis of a plate with mass density $\rho(x, y)$ (mass per unit area). She determines that the desired moment can be calculated by evaluating the integral

$$M_x = \int_{y=0}^{3} \int_{x=y^2}^{9} ye^{-x^2}\, dx\, dy\,.$$

Cynthia correctly notes that the integral can't be evaluated in its present form. She then decides to change the order of integration.

   (a) What function $\rho(x, y)$ describes the density of the material?

   (b) Sketch the region of integration (ie. the shape of the plate) and clearly indicate the axis, boundaries, intersection points, etc.

   (c) Rewrite the integral by switching the order of integration.

   (d) Evaluate M_x using your result from part (c).

2. (25 points) Consider the integral

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \int_{z=-R}^{0} r\ dz\ dr\ d\theta \quad + \quad \int_{\theta=0}^{2\pi} \int_{r=R}^{\sqrt{2}R} \int_{z=-\sqrt{2R^2-r^2}}^{0} r\ dz\ dr\ d\theta\,.$$

   (a) Make a clear sketch of the region of integration in the xyz-coordinate system. (If you have trouble with this, you may "buy" a sketch of the region of integration for 5 points.)

   (b) Express I in cylindrical coordinates using the order $dr\, dz\, d\theta$.

   (c) Express I in spherical coordinates using the order $d\rho\, d\phi\, d\theta$.

   (d) Express I in spherical coordinates using the order $d\phi\, d\rho\, d\theta$.

   (e) Evaluate one of the integrals above to determine the value of I.

3. (25 points) Consider the integral

$$I = \iint_{R_{xy}} e^{xy}\ dx\, dy$$

where R_{xy} is the "diamond" shaped region in the xy-plane bounded by the curves $xy = 1$, $xy = 2$, $x^2y = 2$, and $x^2y = 4$.

   (a) The substitution $u = x^2y$, $v = xy$, simplifies the evaluation of this integral. Find x and y in terms of u and v using the given substitution. Be sure to check this because the rest of the problem depends on this result!

   (b) To evaluate the integral in terms of the new variables, we need to transform the original region R_{xy} into its corresponding region R_{uv} in the uv-plane. Make two clear sketches, one of the original region of integration R_{xy} in the xy-plane, and one of the new region of integration R_{uv} in the uv-plane. Be sure to label all axes, boundaries, intersection points, etc. on each sketch.

   (c) Rewrite the integral I over the region R_{uv} in the uv-plane in terms of u and v.

   (d) Evaluate I in terms of u and v.

4. (25 points) Consider an object moving along a path C in the xy-plane given by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$ for $0 \le t \le 1$ in a force field given by $\mathbf{F} = -2x\,\mathbf{i} + \mathbf{j}$.

   (a) Calculate the work (**flow**) along the path C.

   (b) Calculate the **flux** across the path C.

   (c) If the object moves along the same path but with a different parameterization such that the object moves faster, does the value of the **flux** increase, decrease, or stay the same? Explain your answer.

Projections and distances $\quad \text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Polar coordinates $\quad x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2 \qquad dA = dx\,dy = r\,dr\,d\theta$

Cylindrical and spherical coordinates

Cylindrical to Rectangular	Spherical to Cylindrical	Spherical to Rectangular
$x = r\cos\theta$	$r = \rho\sin\phi$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$z = \rho\cos\phi$	$y = \rho\sin\phi\sin\theta$
$z = z$	$\theta = \theta$	$z = \rho\cos\phi$

$$dV = dx\,dy\,dz = dz\,r\,d\theta\,dr = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Substitutions in multiple integrals

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\left(x(u,v), y(u,v)\right)\,|J(u,v)|\,du\,dv \quad \text{where} \quad J(u,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Mass, moments, and center of mass $\quad$ Mass $\; M = \iint_R \delta\,dA$

Moments $\; M_x = \iint_R y\,\delta\,dA \qquad M_y = \iint_R x\,\delta\,dA \qquad$ Center of mass $\quad \bar{x} = M_y/M \qquad \bar{y} = M_x/M$

Flow and flux $\quad$ Flux $= \int_C \mathbf{F}\cdot\mathbf{n}\,ds \qquad$ Flow $= \int_C \mathbf{F}\cdot\mathbf{T}\,ds$