

**INSTRUCTIONS:** Computers, calculators, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

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1. (30 points) One of the classic Calculus I problems requires one to take a wire of length  $L$  and cut it into two pieces of length  $C$  and  $S$  respectively. The piece of length  $C$  is bent into a circle and the piece of length  $S$  is bent into a square. The objective is to enclose the maximum total area inside the circle and square. *Using Calculus III principles*, determine  $C$  and  $S$ . Clearly explain your approach as you work through the problem! (Hint: you may find it useful to graph the constraint curve in the  $C - S$  plane.)
2. (30 Points) Pete the ant (the same little guy from exam 1) is walking along on a warm flat plate. His path is given by  $\mathbf{r}(t) = 2t^2 \mathbf{i} + t^3 \mathbf{j}$  where distances are measured in millimeters and times are measured in seconds. The temperature distribution on the plate (in  $^{\circ}\text{C}$ ) is given by  $T(x, y) = xy + y^2$ .
  - (a) At what time does Pete cross the point  $A(2, 1)$ ?
  - (b) As Pete is crossing point  $A$ , at what rate is the temperature changing with respect to time (in  $^{\circ}\text{C}/\text{sec}$ )?
  - (c) If Pete continues to walk along path  $C$  for another 0.1 sec, by how much will the temperature change?
  - (d) Based on your result from part (c) above, approximately what is the temperature at his new location?
3. (30 Points) The region inside a large storage container can be described by  $1 \leq x \leq 2$ ,  $1 \leq x^2y \leq 2$ , and  $0 \leq z \leq 1$ , where all lengths are given in meters. There is a lot of dust in the air; in fact the density of dust particles inside the container (particles per cubic meter) is given by  $\rho = x^3y + x^4y^2z$ .
  - (a) Sketch the "footprint" (that is, the floor of the building) in the  $xy$ -plane. Clearly label the boundaries.
  - (b) Write out in detail, but *do not evaluate* the integral required to determine how many dust particles are inside the container.
  - (c) To simplify the integration, you try the following substitution,  $u = x$ ,  $v = x^2y$  and  $w = z$ . Neatly draw the new region of integration in  $uvw$ -space and clearly label the boundaries.
  - (d) Finally, rewrite the integral in part (b) in terms of  $u$ ,  $v$  and  $w$  and evaluate it to determine the number of dust particles in the container.
4. (40 Points) Consider the vector field  $\mathbf{F} = 3x^2y^2 \mathbf{i} + 2x^3y \mathbf{j} + 1 \mathbf{k}$  and the path  $C$  defined by  $\mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{j} + 2 \mathbf{k}$  from the point  $A(0, 0, 2)$  to the point  $B(4, 2, 2)$ . (Note that the path  $C$  is in the plane  $z = 2$ .)
  - (a) At what times are you at the points  $A$  and  $B$ ?
  - (b) Calculate the flow of  $\mathbf{F}$  from point  $A$  to point  $B$  along path  $C$ .
  - (c) If your flow calculation in part (b) can be verified using any theorem(s) from Calculus III, state the theorem(s), and evaluate the appropriate alternate calculation. Otherwise, clearly write "Result cannot be confirmed."
  - (d) Calculate the flux across path  $C$  (in the plane  $z = 2$ ) from point  $A$  to point  $B$ .
  - (e) If your flux calculation in part (d) can be verified using any theorem(s) from Calculus III, state the theorem(s), and evaluate the appropriate alternate calculation. Otherwise, clearly write "Result cannot be confirmed."

5. (40 Points) Consider an object bounded on top by the surface  $z + x^2 + y^2 = 1$  and the bottom by the surface  $z = 0$ .
- Find the parameterization of the curve  $C$  defined by the intersection of the two surfaces.
  - Calculate the circulation of the field given by  $\mathbf{F} = -2y \mathbf{i} + 2x \mathbf{j}$  around the path  $C$  in a counterclockwise direction.
  - If your circulation calculation in part (b) can be verified using any theorem(s) from Calculus III, state the theorem(s), and evaluate the appropriate alternate calculation. Otherwise, clearly write "Result cannot be confirmed."
  - Calculate the outward flux of the (new) field defined by  $\mathbf{G} = 2x \mathbf{i} + 2y \mathbf{j}$  across the top surface of the object.
  - Calculate the outward flux of field  $\mathbf{G}$  across the bottom surface of the object.
  - Based on your results in parts (d) and (e), what is the total outward from the object?
  - If your net outward flux calculation in part (f) can be verified using any theorem(s) from Calculus III, state the theorem(s), and evaluate the appropriate alternate calculation. Otherwise, clearly write "Result cannot be confirmed."

6. (30 Points)

Some physics students purchased a cheap microwave oven, took out the rotating tray and poured a thin, uniform layer of pancake batter across the flat bottom of the microwave oven chamber. After letting the microwave cook the batter, they observed patterns in the batter where the batter overcooked and undercooked. (Believe it or not, by looking at the "hot spots" where the batter overcooked, they could measure the speed of light to within about 1%. Smart folks, these physics students!)

The group modeled the electric field as  $\mathbf{F} = \sin\left(\left(\sqrt{3}\pi/2\right)\sqrt{x^2 + y^2 + z^2}\right) \mathbf{k}$ . The origin is considered to be at the center of the door side at the bottom of the chamber, with  $\mathbf{k}$  pointing up,  $\mathbf{i}$  pointing toward the back of the oven and  $\mathbf{j}$  pointing to the left. Electric fields are conservative, which means that  $\mathbf{F}$  can be obtained from a potential function  $f$  by calculating  $\mathbf{F} = \nabla f$ .

- While letting the batter cook, the students ask you to predict the **shape** of the burn marks in the batter on the floor of the chamber. You can do this by locating the maxima of the potential function  $f$  on the bottom of the chamber ( $z = 0$ ) as a function of  $x$  and  $y$ . Note: you only need to find the shape of the curves in the  $xy$ -plane along which  $f$  has its maximum values. Oh yes, be sure to use Calculus III principles to do so.
- What is the flux of  $\mathbf{F}$  through the front side of the chamber (assuming the door was removed and the dimensions of the opening are 0.5 m across and 0.5 m high)?

**Projections and distances**      $\text{proj}_{\mathbf{A}} \mathbf{B} = \left( \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \right) \mathbf{A}$       $d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$       $d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|$

**Arc length, frenet formulas, and tangential and normal acceleration components**

$$ds = |\mathbf{v}| dt \quad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \quad \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N} \quad \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|1 + (f'(x))^2|^{3/2}} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{|\dot{x}^2 + \dot{y}^2|^{3/2}} \quad \tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

$$\mathbf{a} = a_N \mathbf{N} + a_T \mathbf{T} \quad a_T = \frac{d|\mathbf{v}|}{dt} \quad a_N = \kappa |\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

**The Second Derivative Test**

Suppose  $f(x, y)$  and its first and second partial derivatives are continuous in a disk centered at  $(a, b)$  and  $f_x(a, b) = f_y(a, b) = 0$ . Let  $D = f_{xx}f_{yy} - f_{xy}^2$ .

1. If  $D > 0$  and  $f_{xx} < 0$  at  $(a, b)$ , then  $f$  has a local maximum at  $(a, b)$ .
2. If  $D > 0$  and  $f_{xx} > 0$  at  $(a, b)$ , then  $f$  has a local minimum at  $(a, b)$ .
3. If  $D < 0$  at  $(a, b)$ , then  $f$  has a saddle point at  $(a, b)$ .
4. If  $D = 0$  at  $(a, b)$ , then the test is inconclusive.

**Directional derivative, discriminant, and Lagrange multipliers**

$$\frac{df}{ds} = (\nabla f) \cdot \mathbf{u} \quad f_{xx}f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda \nabla g, \quad g = 0$$

**Taylor's formula** (at the point  $(x_0, y_0)$ )

$$f(x, y) = f(x_0, y_0) + \left[ (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) \right]$$

$$+ \frac{1}{2!} \left[ (x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0)f_{xy}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0) \right]$$

$$+ \frac{1}{3!} \left[ (x - x_0)^3 f_{xxx}(x_0, y_0) + 3(x - x_0)^2(y - y_0)f_{xxy}(x_0, y_0) \right.$$

$$\left. + 3(x - x_0)(y - y_0)^2 f_{xyy}(x_0, y_0) + (y - y_0)^3 f_{yyy}(x_0, y_0) \right] + \dots$$

**Linear approximation error**

$$|E(x, y)| \leq \frac{1}{2} M (|x - x_0| + |y - y_0|)^2, \quad \text{where } \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \leq M$$

**Polar coordinates**      $x = r \cos \theta$       $y = r \sin \theta$       $r^2 = x^2 + y^2$       $dA = dx dy = r dr d\theta$

**Cylindrical and spherical coordinates**

| Cylindrical to Rectangular | Spherical to Cylindrical | Spherical to Rectangular         |
|----------------------------|--------------------------|----------------------------------|
| $x = r \cos \theta$        | $r = \rho \sin \phi$     | $x = \rho \sin \phi \cos \theta$ |
| $y = r \sin \theta$        | $z = \rho \cos \phi$     | $y = \rho \sin \phi \sin \theta$ |
| $z = z$                    | $\theta = \theta$        | $z = \rho \cos \phi$             |

$$dV = dx dy dz = dz r d\theta dr = \rho^2 \sin \phi d\rho d\phi d\theta$$

**Substitutions in multiple integrals**

$$\iint_R f(x, y) dx dy = \iint_G f(x(u, v), y(u, v)) |J(u, v)| du dv \quad \text{where} \quad J(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

**Mass, moments, and center of mass**     Mass      $M = \iint_R \delta dA$

Moments      $M_x = \iint_R y \delta dA$       $M_y = \iint_R x \delta dA$      Center of mass      $\bar{x} = M_y/M$       $\bar{y} = M_x/M$

**Green's Theorem in a plane** (The curve  $C$  is traversed counterclockwise.)

$$\text{Outward Flux} = \oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_R \nabla \cdot \mathbf{F} dA$$

$$\text{Circulation} = \oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} dA$$

**Surface area of level surface**  $f(x, y, z) = c$       $S = \iint_S d\sigma = \iint_S \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA$

**Stoke's Theorem**      $\oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$

**Divergence Theorem of Gauss**      $\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F} dV$