

Be sure to include your name and a grading table on the front of your blue book. You must work all of the problems on this exam. Show ALL of your work and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, a wrong answer with no work will receive no credit, and an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, crib sheets, cell phones, calculators, or electronic devices of any kind are NOT permitted. Please start each problem on a new page. Good luck!

1. (21 points) A large closed rectangular box sits in the first octant with three faces in the coordinate planes. The vertex of this box that does not touch any of the coordinate planes (the one out in space) intersects the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ where a , b , and $c > 0$. Find the position of the vertex that maximizes the volume of the box.

2. (20 points) Loedz is a very creative spider. He built an elaborate web one night in an inspired frenzy which can be modeled as the function $x^2 + 3yy^3$. His less creative (more practical) friend spider Jonesy criticized his design saying that leaves would get caught in the troughs of the thing. Loedz is also a smart spider taking Calc 3. Help Loedz defend his design by computing the following.
 - (a) Identify all critical points of Loedz web.
 - (b) For each critical point in a, identify each as a maximum, a minimum, a saddle point, or inconclusive, showing all work.

3. (20 points) Consider the volume of the solid cut out of the sphere $x^2 + y^2 + z^2 = 25$ by the cylinder $x^2 + y^2 = 9$ above the $x - y$ plane.
 - (a) Set up **but do not evaluate** the integral of the volume in the form $V = \iint f(x, y) dx dy$ by including the correct function f and limits of integration.
 - (b) Transform the integral in part (a) into polar coordinates **but do not evaluate**.
 - (c) Transform the integral in (a) to a volume triple integral of the form $V = \iiint dz dx dy$ **but do not evaluate**.
 - (d) Pick *one* of the integrals in parts a-c and evaluate it.

4. (18 points) Three unrelated chain rule problems.

- (a) Verify that $u(x, t) = f(x + ct) + g(x - ct)$ where c is a constant and $r = x + ct$ and $s = x - ct$ satisfies the wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.
- (b) Let $t = f(u, v)$ where $u = u(x, y, z, w)$ and $v = v(x, y, z, w)$. Write out the change in t with respect to y **and** justify whether it should be $\frac{\partial t}{\partial y}$ or $\frac{dt}{dy}$.
- (c) Express x and y in polar coordinates to show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

Hint: You will need to use the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

5. (21 points) A hiker is on a mountain given by the function $z = f(x, y) = 106 - x^2 - y^2$ and is standing at the point the point $(3, 4)$.

- (a) A thunderstorm is beginning to form, and knowing that lightning is the second leading cause of weather-related death in Colorado, the hiker wants to get down from the mountain as quickly as possible. In which *direction* should she hike to *descend* the mountain the fastest?
- (b) Find the first order Taylor series approximation for $f(x, y)$ at the point $(3, 4)$.
- (c) Assume the density of the mountain is a constant ($\rho = a$) and that the base of the mountain is the region $0 \leq x \leq 1$ and $0 \leq y \leq 2$. The equation for finding the mass of the mountain is given by $\iiint_D \rho dV$. Find the mass of the mountain.

Projections, and distances from a point to a line and a plane

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \right) \mathbf{A} \quad d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \quad d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|$$

Arc length, Frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}| dt \quad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \quad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \quad \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \quad a_T = \frac{d|\mathbf{v}|}{dt} \quad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, Discriminant, and Lagrange Multipliers

$$\frac{df}{ds} = (\nabla f) \cdot \mathbf{u} \quad f_{xx}f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda \nabla g, \quad g = 0$$

Taylor's Formula (at the point (x_0, y_0))

$$\begin{aligned} f(x, y) = & f(x_0, y_0) + \left((x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) \right) \\ & + \frac{1}{2!} \left((x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0)f_{xy}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0) \right) \\ & + \frac{1}{3!} \left((x - x_0)^3 f_{xxx}(x_0, y_0) + 3(x - x_0)^2(y - y_0)f_{xxy}(x_0, y_0) \right. \\ & \left. + 3(x - x_0)(y - y_0)^2 f_{xyy}(x_0, y_0) + (y - y_0)^3 f_{yyy}(x_0, y_0) \right) + \dots \end{aligned}$$

Linear Approximation Error, $E(x, y)$

$$|E(x, y)| \leq \frac{1}{2}M(|x - x_0| + |y - y_0|)^2, \quad \text{where } \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \leq M$$

Polar coordinates $x = r \cos \theta$ $y = r \sin \theta$ $r^2 = x^2 + y^2$

$$dA = dx dy = r dr d\theta$$