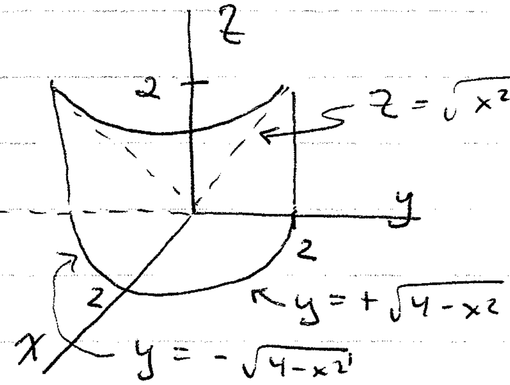
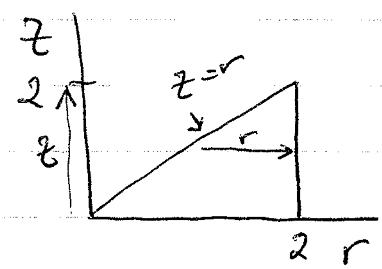
1a $\displaystyle\int_{x=0}^{2}\int_{y=-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\int_{z=0}^{\sqrt{x^2+y^2}} dz\,dy\,dx$




$x^2+y^2=z^2=r^2=4$

$z=2$

b. $\theta$ goes from $-\pi/2$ to $\pi/2$, then in 2D

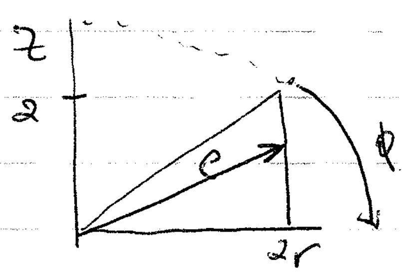



r goes from line $r=z$ to $r=2$

z goes from 0 to $z_{max}=2$

$$\int_{\theta=-\pi/2}^{\pi/2}\int_{z=0}^{2}\int_{r=z}^{2} r\,dr\,dz\,d\theta$$

c. $\theta$ is same as cylindrical, in 2D




$\phi$ goes from $\pi/4$ to max $\pi/2$

$\rho$ goes from 0 to line

line is $r=2=\rho\sin\phi$

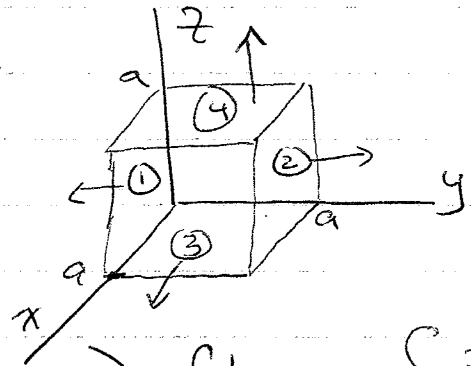
so $\rho=\dfrac{2}{\sin\phi}$ = line

$$\int_{\theta=-\pi/2}^{\pi/2}\int_{\phi=\pi/4}^{\pi/2}\int_{\rho=0}^{2/\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

d. solve b $=\displaystyle\int_{z=0}^{2}\frac{r^2}{2}\Big|_{r=z}^{2}\ \theta\Big|_{-\pi/2}^{\pi/2}=\frac{\pi}{2}\int_{z=0}^{2}(4-z^2)\,dz$

$=\dfrac{\pi}{2}\left(4z-\dfrac{z^3}{3}\right)\Big|_{z=0}^{2}=\dfrac{\pi}{2}\left(8-\dfrac{8}{3}\right)=\dfrac{\pi}{2}\cdot 8\cdot\dfrac{2}{3}=\boxed{8\pi/3}$

2.




$\vec{F} = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$

expecting positive flux and no curl based on $\vec{F}$

a) flux $\int \vec{F}\cdot\hat{n}\,d\sigma$

for side ① in x-z plane, $\hat{n} = -\hat{y}$

$g = x = 0 \quad \nabla g = \hat{\imath} \quad |\nabla g| = 1$

$\vec{F}\cdot\hat{n} = -y^2$

$\int \vec{F}\cdot\hat{n}\,dA = \int -y^2\,dx\,dz$ but on x-z plane $y=0$ so flux $= 0$

note that for all the sides that are coordinate planes $\vec{F} = 0$ and there is no flux

② $\hat{n} = \hat{y}$, $g = x = a \quad \nabla g = \hat{\imath} \quad |\nabla g| = 1$

$\vec{F}\cdot\hat{n} = +y^2 \quad \int \vec{F}\cdot\hat{n}\,d\sigma = \int y^2\,dx\,dz$

$y^2 = a^2$ on this surface

integral $= a^2 \int dx\,dz = a^2$ (area of side) $= a^4$

③ $\hat{n} = \hat{x}$, $\vec{F}\cdot\hat{n} = x^2 = a^2 \quad \int \vec{F}\cdot\hat{n}\,d\sigma = a^2\cdot a^2 = a^4$

④ by symmetry $= a^4$

total flux $= 3a^4$ (pos. flux :))

b) Divergence Theorem $\iint \vec{F}\cdot\hat{n}\,d\sigma = \iiint \nabla\cdot\vec{F}\,dV$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2)\cancel{\hat{x}} + \frac{\partial}{\partial y}(y^2)\cancel{\hat{y}} + \frac{\partial}{\partial z}(z^2)\cancel{\hat{z}} = 2x\cancel{\hat{x}} + 2y\cancel{\hat{y}} + 2z\cancel{\hat{z}}$$

$$\int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a} (2x+2y+2z)\,dz\,dy\,dx = 2\iint_{x\,y} (x+y)z + \frac{z^2}{2}\Big|_{z=0}^{a}\,dy\,dx$$

$$= 2\iint_{x\,y}\left(a(x+y) + \frac{a^2}{2}\right)dy\,dx = 2a\int_x \left(x + \frac{a}{2}\right)y + \frac{y^2}{2}\Big|_{y=0}^{a}\,dx$$

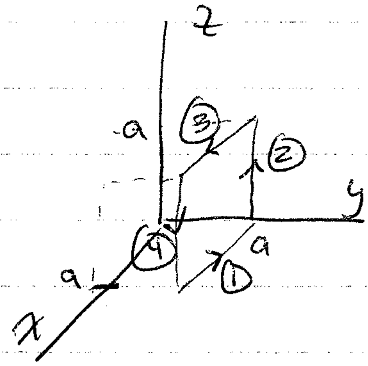
2b cont.

$$= 2a \int_{x=0}^{a} \left(x + \frac{a}{2}\right) a + \frac{a^2}{2} \, dx$$

$$= 2a^2 \int_{x=0}^{a} x + a \, dx = 2a^2 \left(\frac{x^2}{2} + ax\right)\Big|_{x=0}^{a}$$

$$= 2a^2\left(\frac{a^2}{2} + a^2\right) = a^4 + 2a^4 = \boxed{3a^4} \checkmark$$

c. right panel surface, direct calc




$$\int \vec{F} \cdot dr = \int \vec{F} \cdot \frac{dr}{dt} \, dt$$

① $\vec{r}$ goes from $(a, a, 0)$ to $(0, a, 0)$

$$\vec{r} = a(1-t)\hat{\imath} + a\hat{\jmath} + 0\hat{k} \quad t = 0 \to 1$$

$$\frac{d\vec{r}}{dt} = -a\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

$$\int \vec{F} \cdot \frac{d\vec{r}}{dt} \, dt = \int (x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}) \cdot (-a\hat{\imath}) \, dt$$

$$= \int_{t=0}^{1} x^2(-a) \, dt = \int_{t=0}^{1} a^2(1-t)^2(-a) \, dt = -a^3 \int 1 - 2t + t^2 \, dt$$

$$= -a^3 \left(t - \frac{2t^2}{2} + \frac{t^3}{3}\right)\Big|_0^1 = -a^3\left(1 - 1 + \frac{1}{3}\right) = \boxed{-a^3/3 = ①}$$

③ $\vec{r} = at\hat{\imath} + a\hat{\jmath} + a\hat{k} \quad t = 0 \to 1 \quad (0, a, a)$ to $(a, a, a)$

$$\frac{d\vec{r}}{dt} = a\hat{\imath} \qquad \int \vec{F} \cdot \frac{d\vec{r}}{dt} \, dt = \int (x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}) \cdot (a\hat{\imath}) \, dt$$

$$= a \int x^2 \, dt = a \int_{t=0}^{1} a^2 t^2 \, dt = a^3 \frac{t^3}{3}\Big|_0^1 = \boxed{+\frac{a^3}{3} = ③}$$

2c cont.

(2) $(0,a,0)$ to $(0,a,a)$ $\vec{r} = 0\hat{\imath} + a\hat{\jmath} + at\hat{k}$

$t=0$ to $1$

$$\frac{d\vec{r}}{dt} = a\hat{k} \quad \int \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int_{t=0}^{1} (x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k})\cdot(a\hat{k})\,dt$$

$$= a\int_{t=0}^{1} z^2\,dt = a^2\int a^2t^2\,dt = a^3\frac{t^3}{3}\Big|_0^1 = \boxed{\frac{a^3}{3} = (2)}$$

(4) $(a,a,a)$ to $(a,a,0)$ $\vec{r} = a\hat{\imath} + a\hat{\jmath} + (1-t)a\hat{k}$ $t = 0$ to $1$

$$\frac{d\vec{r}}{dt} = -a\hat{k} \quad \int \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int (x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k})\cdot(-a\hat{k})\,dt$$

$$= -a\int z^2\,dt = -a\int_{t=0}^{1} (1-t)^2a^2\,dt = -a^3\int (1-2t+t^2)\,dt$$

$$= -a^3\left(t - \frac{2t^2}{2} + \frac{t^3}{3}\right)\Big|_{t=0}^{1} = -a^3\left(1 - 1 + \frac{1}{3}\right) = \boxed{-\frac{a^3}{3} = (4)}$$

$\Sigma$ (1) + (2) + (3) + (4) $= \boxed{0}$ net circulation (as expected)

d. Stoke's theorem (technically in 3D)

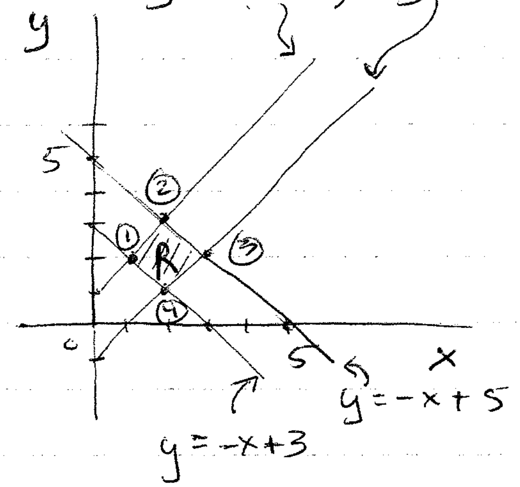
$$\int \vec{F}\cdot d\vec{r} = \iint \nabla\times\vec{F}\cdot\hat{n}\,d\sigma$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y}z^2 - \frac{\partial}{\partial z}y^2\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}z^2 - \frac{\partial}{\partial z}x^2\right) + \hat{k}\left(\frac{\partial}{\partial x}y^2 - \frac{\partial}{\partial y}x^2\right) = 0$$

so curl $= 0$

$\Rightarrow$ circulation $= \boxed{0}$ (as expected)

3 a) $y = x+1$, $y = x-1$, $y = -x+5$, $y = -x+3$

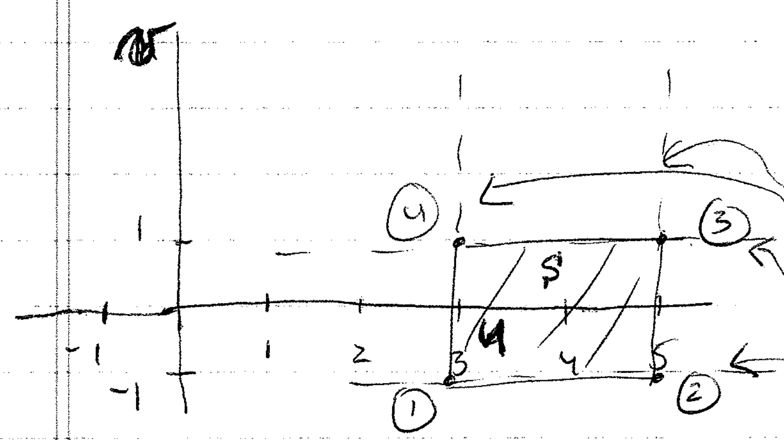



b) $x = \frac{1}{2}(u+v)$ $\quad -\left(y = \frac{1}{2}(u-v)\right)$

$\underline{y = \frac{1}{2}(u-v)}$ $\quad \underline{x = \frac{1}{2}(u+v)}$

$x+y = u$ $\quad x-y = v$

1. $x=1, y=2 \Rightarrow u = x+y = 1+2 = \boxed{3=u}$, $v = x-y = 1-2 = \boxed{-1=v}$
2. $x=2, y=3 \Rightarrow u = 2+3 = \boxed{5=u}$, $v = 2-3 = \boxed{-1=v}$
3. $x=3, y=2 \Rightarrow u = 2+3 = \boxed{5=u}$, $v = 3-2 = \boxed{1=v}$
4. $x=2, y=1 \Rightarrow u = 2+1 = \boxed{3=u}$, $v = 2-1 = \boxed{1=v}$




equations

$y-x=1 \Rightarrow x-y = \boxed{-1=v}$

$y-x=1 \Rightarrow x-y = \boxed{1=v}$

$y+x = \boxed{5=u}$

$y+x = \boxed{3=u}$

c) $\iint e^{x+y}\,dx\,dy = \int_{u=3}^{5}\int_{v=-1}^{1} |J| e^{u}\,du\,dv$

3c cont.

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$$

$$\int_{u=3}^{5} \int_{v=-1}^{1} \frac{1}{2} e^{u} \, du \, dv$$

$$= \frac{1}{2} v \Big|_{-1}^{1} e^{u} \Big|_{3}^{5} = \frac{1}{2}(2)(e^5 - e^3) = e^5 - e^3$$

Hee Hee

4a $\nabla \times \vec{F} \overset{?}{=} 0 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+4zx & 2y(x+z) & y^2+2x^2 \end{vmatrix}$

$= \hat{\imath}(2y-2y) - \hat{\jmath}(4x-4x) + \vec{k}(2y-2y) \overset{\checkmark}{=} 0$

Yes, conservative

b) potential

$\frac{\partial f}{\partial z} = y^2 + 2x^2 \qquad f = (y^2+2x^2)z + g(x,y)$

$\frac{\partial f}{\partial x} = y^2 + 4\cancel{z}x = \frac{\partial}{\partial x}((y^2+2x^2)z + g)$

$y^2 + 4zx = 4xz + \frac{\partial g}{\partial x}$

$g = y^2 x + h(y)$

$\frac{\partial f}{\partial y} = 2y(x+z) = \frac{\partial}{\partial y}\left[(y^2+2x^2)z + y^2x + h\right]$

$2yz + 2yx = 2yz + 2yx + h' \qquad h = 0$

then $\boxed{f = (y^2+2x^2)z + y^2 x + c}$

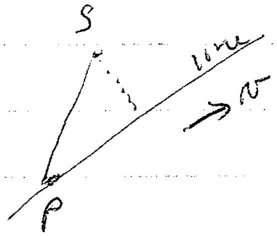
check $\vec{F} \overset{?}{=} \nabla f = (4xz + y^2)\hat{\imath} + (2yz + 2yx)\hat{\jmath}$
$+ (y^2+2x^2)\hat{k}$ ✓

c) The Hitchhiker's Guide to the Galaxy

5. $w = x^2 y^3 z^4 \quad x = 2t+1,\ y = 3t+2,\ z = 5t+4$

a) $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$
$$= (2xy^3z^4)(2) + (3x^2y^2z^4)(3) + (4x^2y^3z^3)(5)$$

b)




$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} \qquad \vec{v} = 2\hat{i} + 3\hat{j} + 5\hat{k}$$
$$|\vec{v}| = \sqrt{4+9+25} = \sqrt{13+25} = \sqrt{38}$$

$\vec{PS}$ need a point P on line
choose $t = 0 \Rightarrow (1, 2, 4)$

$$\vec{PS} = (1-0)\hat{i} + (2-0)\hat{j} + (4-0)\vec{k} = \hat{i} + 2\hat{j} + 4\vec{k}$$
$$d = |(\hat{i} * 2\hat{j} + 4\vec{k}) \times (2\hat{i} + 3\hat{j} + 5\vec{k})| / \sqrt{38}$$

$$\vec{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \vec{k} \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{vmatrix} = \hat{i}(10-12) - \hat{j}(5-8) + \vec{k}(3-4) = -2\hat{i} + 3\hat{j} - \vec{k}$$

$$|\vec{PS} \times \vec{v}| = \sqrt{4+9+1} = \sqrt{14}$$

$$\frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{14}}{\sqrt{38}} = \boxed{\frac{\sqrt{7}}{\sqrt{19}}}$$

c) vector is $+2\hat{i} + 3\hat{j} + 5\vec{k} \Rightarrow -2x + 3y - z = c$
point is $(0,0,0) \Rightarrow 0 + 0 + 0 = c$
$$\boxed{+2x + 3y + 5z = 0}$$

5d directional derivative

$$\nabla f\big|_{P_0} \cdot \hat{u} = \left(2xy^3z^4\,\hat{i} + 3x^2y^2z^4\,\hat{j} + 4x^2y^3z^3\,\vec{k}\right)\Big|_{P_0} \cdot \hat{u}$$

$$\hat{u} = \frac{2\hat{i} + 3\hat{j} + 5\vec{k}}{\sqrt{38}}$$

$$= \left(-2\hat{i} + 3\hat{j} - 4\vec{k}\right) \cdot \left(2\hat{i} + 3\hat{j} + 5\vec{k}\right) / \sqrt{38}$$

$$= (-4 + 9 - 20)/\sqrt{38}$$

$$= \boxed{-15/\sqrt{38}}$$