

INSTRUCTIONS: Computers, calculators, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (20 points) Using an appropriate technique from Calculus III, determine the extreme values of the function $f(x, y) = x^2 + \frac{y^2}{4}$, assuming that $xy = 1$. For each point you find, be sure to state whether the function has a local maxima or minima at that location, and the value of $f(x, y)$.
2. (20 points) The surface elevation, z , of a small mountain area that you have just purchased is given by the function $z = -y^2 - x^4 + 4x^2 + 54$ for $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.
 - (a) Where on this property would you build a hut? (Hint: perhaps a level spot that is not a summit. Clearly explain your reasoning, and show your calculations, to support your answer.)
 - (b) A stream emerges from a spot on the surface located at $P_0(0, 1, 53)$ and flows downhill following the line of steepest descent. At the point P_0 , in which direction does the stream flow? (Give your answer as a vector with \mathbf{i} , \mathbf{j} , and \mathbf{k} components.)
 - (c) By approximately how much has the water in the stream descended after it has travelled a short distance $s = 0.2$ from the source P_0 ?
3. (20 points) A hummingbird is flying along the path $\mathbf{r}(t)$ during the evening when the temperature distribution is given by the function $T(x, y, z)$. At some time t^* (and only at this particular time), you know that the hummingbird's position, velocity and acceleration are $\mathbf{r}(t^*) = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, $\mathbf{v}(t^*) = 2\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, and $\mathbf{a}(t^*) = 4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ respectively. Furthermore, you know that $\nabla T|_{(2,4,3)} = 2\mathbf{i} + 1\mathbf{j} + 2\mathbf{k}$, and $T(2, 4, 3) = 42$.
 - (a) As the hummingbird flies past the location $\mathbf{r}(t^*)$ at what rate is the temperature T changing with respect to distance?
 - (b) As it flies past location $\mathbf{r}(t^*)$, at what rate is the temperature T changing with respect to time?
 - (c) If the hummingbird continues on its original path $\mathbf{r}(t)$ for a short interval of time $\Delta t = 0.1$, by approximately how much does the temperature change?
 - (d) However, suppose at time t^* something suddenly scares the hummingbird and it starts to fly in a direction that happens to be the direction of the greatest rate of decrease of T . Assuming the hummingbird maintains its same speed, by approximately how much does the temperature change if it flies for $\Delta t = 0.1$?
4. (20 points) Consider the function $f(x, y) = x + y + \cos(x + y)$.
 - (a) Calculate the *second order* Taylor approximation to $f(x, y)$ near the *origin*.
 - (b) Now, calculate the *first order* approximation to $f(x, y)$ near the point P located at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$,
 - (c) Estimate the error associated with your linearization in part (b) assuming that you only use it for values of x and y such that $\left|x - \frac{\pi}{4}\right| \leq 0.2$ and $\left|y - \frac{\pi}{4}\right| \leq 0.2$.

OVER

5. (20 points)

It is suspected that a local bluebook manufacturing plant is emitting trioxin. Some local university students decide to measure the trioxin levels, but they don't have an instrument to directly measure the trioxin concentration. Having done well in their chemistry classes, they know that trioxin levels, C , can be determined from $C = \frac{xz}{y^2}$, where x , y , and z are the concentrations of CKW, chloride, and blue dye #42, respectively. The students can easily take measurements of these chemicals using some old equipment they found in their chemistry lab.

The students sneak out one night and take measurements. Unfortunately, all of their measurements are exactly 20% high. News of their measurements and trioxin concentration calculations were reported, and then heavily criticized, in the local newspapers because their measurements of x , y , and z were not correct.

- By what percentage is their trioxin concentration value off?
- Is their trioxin concentration calculation high or low?
- Based on your calculations in parts (a) and (b) explain why, or why not, the criticisms in the newspaper would be justified.

Projections and distances

$$\text{proj}_{\mathbf{A}} \mathbf{B} = \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \right) \mathbf{A} \quad d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \quad d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|$$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}| dt \quad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \quad \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N} \quad \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|1 + (f'(x))^2|^{3/2}} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{|\dot{x}^2 + \dot{y}^2|^{3/2}} \quad \tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

$$\mathbf{a} = a_N \mathbf{N} + a_T \mathbf{T} \quad a_T = \frac{d|\mathbf{v}|}{dt} \quad a_N = \kappa |\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f) \cdot \mathbf{u} \quad f_{xx}f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda \nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$f(x, y) = f(x_0, y_0) + \left[(x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) \right]$$

$$+ \frac{1}{2!} \left[(x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0)f_{xy}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0) \right]$$

$$+ \frac{1}{3!} \left[(x - x_0)^3 f_{xxx}(x_0, y_0) + 3(x - x_0)^2(y - y_0)f_{xxy}(x_0, y_0) \right.$$

$$\left. + 3(x - x_0)(y - y_0)^2 f_{xyy}(x_0, y_0) + (y - y_0)^3 f_{yyy}(x_0, y_0) \right] + \dots$$

Linear approximation error

$$|E(x, y)| \leq \frac{1}{2} M (|x - x_0| + |y - y_0|)^2, \quad \text{where } \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \leq M$$