1. Solution

$\nabla f = 2x\,i + \frac{1}{2} y\,j$

$\nabla g = y\,i + x\,j$

$\nabla f = \lambda \nabla g$

$2x\,i + \frac{1}{2} y\,j = \lambda y\,i + \lambda x\,j$

$\begin{cases} 2x = \lambda y \\ \frac{1}{2} y = \lambda x \end{cases} \Rightarrow x^2 = \lambda^2 x \Rightarrow x = 0$ or $\lambda = \pm 1$

since $xy = 1 \Rightarrow x$ cannot be 0

case 1 : $\lambda = -1$ $\quad y = -2x \Rightarrow -2x^2 = 1$ no solution

case 2 : $\lambda = 1$ $\quad y = 2x \Rightarrow 2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$

$\therefore \begin{cases} x = \frac{1}{\sqrt{2}} \\ y = \sqrt{2} \end{cases}$ or $\begin{cases} x = \frac{-1}{\sqrt{2}} \\ y = -\sqrt{2} \end{cases}$

$f_x = 2x \qquad f_{xx} = 2$

$f_y = \frac{1}{2} y \qquad f_{yy} = \frac{1}{2} \Rightarrow f_{xx} f_{yy} - f_{xy}^2 = 1 > 0$

$f_{xy} = 0 \qquad f_{xx} > 0$

$\Rightarrow$ local mininum at $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$

$f\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right) = f\left(\frac{-1}{\sqrt{2}}, \sqrt{2}\right) = 1$

Problem 2

Idea

$-x^2(x^2-4)$ — x profile

$-y^2$ — y profile

$+\sqrt{4}$ to get ≥ 0

$[-3,3]^2$

(a)

$\frac{df}{dx} = -4x^3+8x=0$

$-4x(x^2-2)=0$

$x_1=0$ or $x_{2/3}=\pm\sqrt{2}$

$\frac{df}{dy} = 2y=0 \Rightarrow y=0$ $(x,y)=(0,0)$ saddle pt

$\rightarrow P_1(-\sqrt{2},0)$, $P_2(0,0)$, $P_3(\sqrt{2},0)$ are extrema

max saddle max

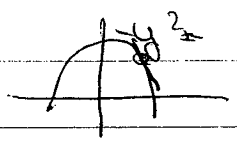
for argumentation that $P_2(0,0)$ is a saddle (various ways acceptable).

(b) $\text{grad} f = \left(\frac{df}{dx}, \frac{df}{dy}\right) = \begin{pmatrix} -4x^3+8x \\ 2y \end{pmatrix}_P$

Surface is : $\{(x,y,z) \mid f(x,y)=z\}$

Notice that the fcten is the add. sum of (x-profile) ⊕ (y-prof)

$\rightarrow$ downhill dir is $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$



Profile $(0 \ 1 \ -2)$

If people write $-\text{grad} f = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

(c) ~~$|\nabla f|\, ds$~~

$\Delta H \rightarrow (\Delta\text{length})$: $|\nabla f|\, ds = \left|\begin{pmatrix} 0 \\ 2 \end{pmatrix}\right| \cdot 0.2 = 2 \cdot 0.2 = 0.4$

3. $r(t^*) = 2\hat{i} + 4\hat{j} + 3\hat{k}$ $\qquad \nabla T\big|_{(2,4,3)} = 2\hat{i} + \hat{j} + 2\hat{k}$

$v(t^*) = 2\hat{i} + 2\hat{j} + 5\hat{k}$ $\qquad T(2,4,3) = 42$

$a(t^*) = 4\hat{i} + 3\hat{j} + 4\hat{k}$

a. $$\frac{dT}{ds} = \nabla T \cdot u = \nabla T \cdot \frac{v}{|v|} = (2\hat{i} + \hat{j} + 2\hat{k}) \cdot \frac{(2\hat{i} + 2\hat{j} + 5\hat{k})}{\sqrt{33}}$$

$$= \frac{1}{\sqrt{33}}(4+2+10) = \boxed{\frac{16}{\sqrt{33}}}$$

b. $$\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt} = \frac{dT}{ds} |v| = \frac{16}{\sqrt{33}} \cdot \sqrt{33} = \boxed{16}$$

OR

$$= \nabla T \cdot v = (2\hat{i} + \hat{j} + 2\hat{k}) \cdot (2\hat{i} + 2\hat{j} + 5\hat{k}) = 4 + 2 + 10 = \boxed{16}$$

c. $$\Delta T \approx \frac{dT}{dt} \Delta t = 16(0.1) = \boxed{1.6}$$

d. New direction: $$\frac{dT}{ds} = \left(\nabla T \cdot \frac{-\nabla T}{|\nabla T|}\right) = -\frac{|\nabla T|^2}{|\nabla T|} = -|\nabla T| = -3$$

So $$\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt} = -3|v| = -3\sqrt{33}$$

So $$\Delta T \approx \left(\frac{dT}{dt}\right) \Delta t = (-3\sqrt{33})(0.1) = \boxed{-0.3\sqrt{33}}$$

4\. a) $f(0,0) = \cos(0) = 1$

$f_x = 1 - \sin(x+y) \Rightarrow f_x(0,0) = 1$

$f_y = 1 - \sin(x+y) \Rightarrow f_y(0,0) = 1$

$f_{xx} = -\cos(x+y) \Rightarrow f_{xx}(0,0) = -1$

$f_{yy} = -\cos(x+y) \Rightarrow f_{yy}(0,0) = -1$

$f_{xy} = -\cos(x+y) \Rightarrow f_{xy}(0,0) = -1$

$$\Rightarrow f(x,y) = 1 + [x + y] + \frac{1}{2!}\left[x^2(-1) + 2xy(-1) + y^2(-1)\right]$$

$$\Rightarrow \boxed{f(x,y) = 1 + x + y + \tfrac{1}{2}\left(-x^2 - 2xy - y^2\right)}$$

b) $f(\pi/4, \pi/4) = \pi/2$

$f_x(\pi/4, \pi/4) = 0 \qquad \Rightarrow f(x,y) = \pi/2 + [x(0) + y(0)]$

$f_y(\pi/4, \pi/4) = 0$

$$\Rightarrow \boxed{f(x,y) = \pi/2}$$

c) $|E(x,y)| \le \frac{1}{2} M \left(\overset{\le 0.2}{|x - x_0|} + \overset{\le 0.2}{|y - y_0|}\right)^2$

$M = \max(|f_{xx}|, |f_{yy}|, |f_{zz}|)$

$-\cos(x+y)$ is bounded by 1! $\Rightarrow M = 1$

$\Rightarrow |E(x,y)| \le \frac{1}{2}(0.4)^2 \Rightarrow \boxed{|E(x,y)| \le 0.08}$

OR

$$E(x,y) = \frac{1}{2!}\Big(\underbrace{(x-\pi/4)}_{\le 0.2}{}^2 \cancelto{1}{f_{xx}(\pi/4,\pi/4)} + 2\overset{\le 0.2}{(x-\pi/4)}\overset{\le 0.2}{(y-\pi/4)} \cancelto{1}{f_{xy}(\pi/4,\pi/4)}$$
$$+ \underbrace{(y-\pi/4)}_{\le 0.2}{}^2 \cancelto{1}{f_{yy}(\pi/4,\pi/4)}\Big)$$

$$\Rightarrow E(x,y) = \tfrac{1}{2}\left((0.2)^2 + 2(0.2)(0.2) + (0.2)^2\right) = \tfrac{1}{2}(0.04 + 0.08 + 0.04)$$

$$\Rightarrow \boxed{E(x,y) = 0.08}$$

⑤ Solution

$C = \frac{xz}{y^2}$ all measurements (x, y, z) are reported 20% high

) (a) By what % is their trioxin concentration off?

2 methods to calculate this!

#1: $dC = C_x\,dx + C_y\,dy + C_z\,dz$

$C_x = \frac{z}{y^2} \quad C_y = -\frac{2xz}{y^3} \quad C_z = \frac{x}{y^2}$

since we want % C...

$$\frac{dC}{C} = \frac{\cancel{z}}{\cancel{y^2}} \cdot \frac{\cancel{y^2}}{x\cancel{z}}\,dx - \frac{2\cancel{x}\cancel{z}}{y^{\cancel{3}}} \frac{\cancel{y^2}}{\cancel{xz}}\,dy + \frac{\cancel{x}}{\cancel{y^2}} \frac{\cancel{y^2}}{\cancel{x}z}\,dz$$

$$\frac{dC}{C} = \frac{dx}{x} - \frac{2dy}{y} + \frac{dz}{z}$$

now, $\frac{dz}{z} = \frac{dy}{y} = \frac{dx}{x} = 0.2$

$$\frac{dC}{C} = 0.2 - 2(0.2) + 0.2 = 0$$

$\boxed{0\% \text{ off}}$

#2 $C = \frac{xz}{y^2}$ 20% high means $C = \frac{(x+0.2x)(z+0.2z)}{(y+0.2y)^2}$

$$= \frac{(1.2x)(1.2z)}{(1.2y)^2}$$

$$= \frac{\cancel{(1.2)^2}\,xz}{\cancel{(1.2)^2}\,y^2} = \frac{xz}{y^2}$$

so even with 20% error, C is the same!

$\boxed{0\% \text{ off}}$

(5) solution

(b) Is their trioxin concentration high or low?

Neither! It's the same.

(c) Are the criticisms justified?

- yes, the students' measurements were 20% off! They got <u>lucky</u> that their system was such that the error balanced...

- yes, the students were sneaking out and taking measurements without permission.

- no, the measurement error was + 20%, but the reported trioxin levels were ok, so no need to criticize!

(reasonable and articulate answers were accepted)