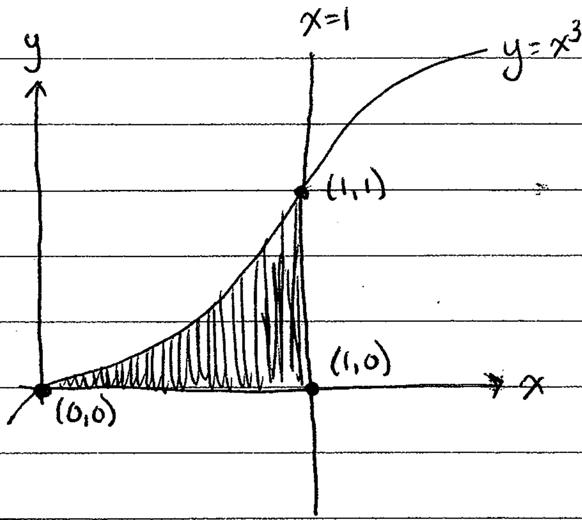
Exam - #3 → Problem #1

ⓐ $\boxed{f(x,y) = \dfrac{\sin(\pi x^2)}{y x^2}}$

ⓑ



for graph
for boundary labels
for axes labels
for intersections labelled

ⓒ $$\int_0^1 \int_0^{x^3} \frac{\sin(\pi x^2)}{x^2}\, dy\, dx$$

for limits
for integrand
for dydx

ⓓ $$\int_0^1 \int_0^{x^3} \frac{\sin(\pi x^2)}{x^2}\, dy\, dx = \int_0^1 \frac{x^3 \sin(\pi x^2)}{x^2}\, dx = \int_0^1 x \sin(\pi x^2)\, dx$$

for first evaluation
for second evaluation
for plugging in limits

$x = 0 \Rightarrow u = 0$ and $x = 1 \Rightarrow u = 1$

$u = x^2 \Rightarrow du = 2x\, dx \Rightarrow \dfrac{du}{2} = x\, dx$

$$= \tfrac{1}{2} \int_0^1 \sin(\pi u)\, du = -\tfrac{1}{2} \frac{\cos(\pi u)}{\pi} \Big|_0^1 = -\tfrac{1}{2\pi}\left(\cos(\pi) - \cos(0)\right)$$

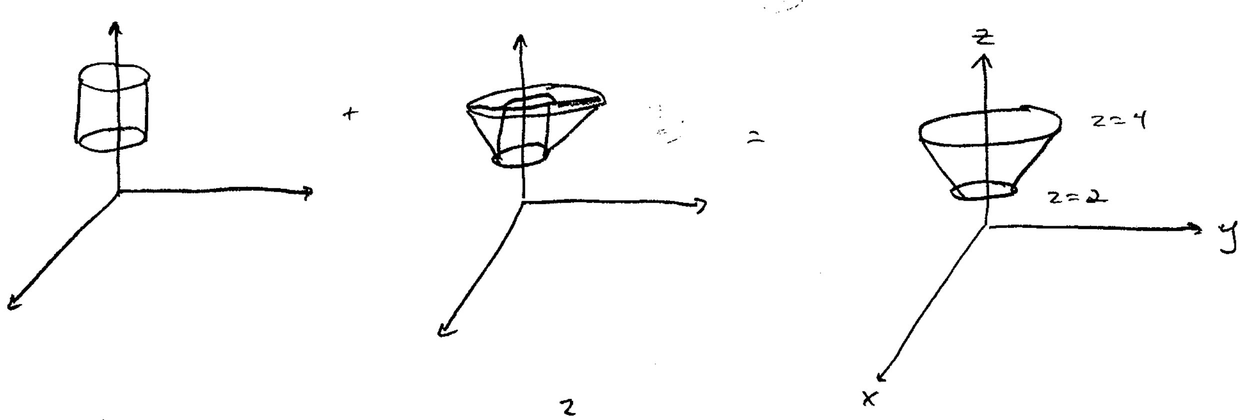
$$= -\tfrac{1}{2\pi}(-1 - 1) = \boxed{\tfrac{1}{\pi}}$$

(2)

$$I = \int_0^{2\pi}\int_0^{2}\int_2^{4} r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_2^{4}\int_r^{4} r\,dz\,dr\,d\theta$$

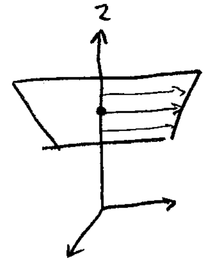
$\theta \quad r \quad z \qquad\qquad \theta \quad r \quad z$

a)



b)

$$I = \int_0^{2\pi}\int_2^{4}\int_0^{z} r\,dr\,dz\,d\theta$$



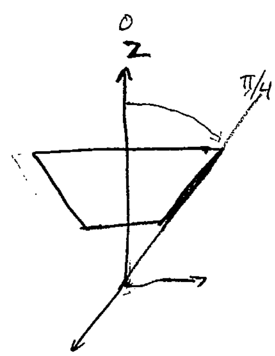
c)

$$I = \int_0^{2\pi}\int_0^{\pi/4}\int_{2\sec\phi}^{4\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

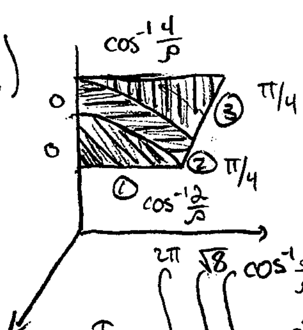
$z = \rho\cos\phi$

$z = 2$ to $z = 4$

so $\rho = 2\sec\phi$ to $\rho = 4\sec\phi$



d)



$$I = \int_0^{2\pi}\int_2^{\sqrt{8}}\int_0^{\cos^{-1}\frac{2}{\rho}} \rho^2\sin\phi\,d\phi\,d\rho\,d\theta + \int_0^{2\pi}\int_{\sqrt{8}}^{4}\int_0^{\pi/4} \rho^2\sin\phi\,d\phi\,d\rho\,d\theta + \int_0^{2\pi}\int_4^{\sqrt{32}}\int_{\cos^{-1}\frac{4}{\rho}}^{\pi/4} \rho^2\sin\phi\,d\phi\,d\rho\,d\theta$$

(1) (2) (3)

e) evaluate (b) $\int_0^{2\pi}\int_2^{4} \frac{z^2}{2} = \int_0^{2\pi} \frac{64-8}{6} = \boxed{\frac{\pi\cdot 56}{3}}$

3

a. $3u = -y + x$

$+\quad 9v = 4y - x$

$3u + 9v = 3y$

$\boxed{u + 3v = y}$

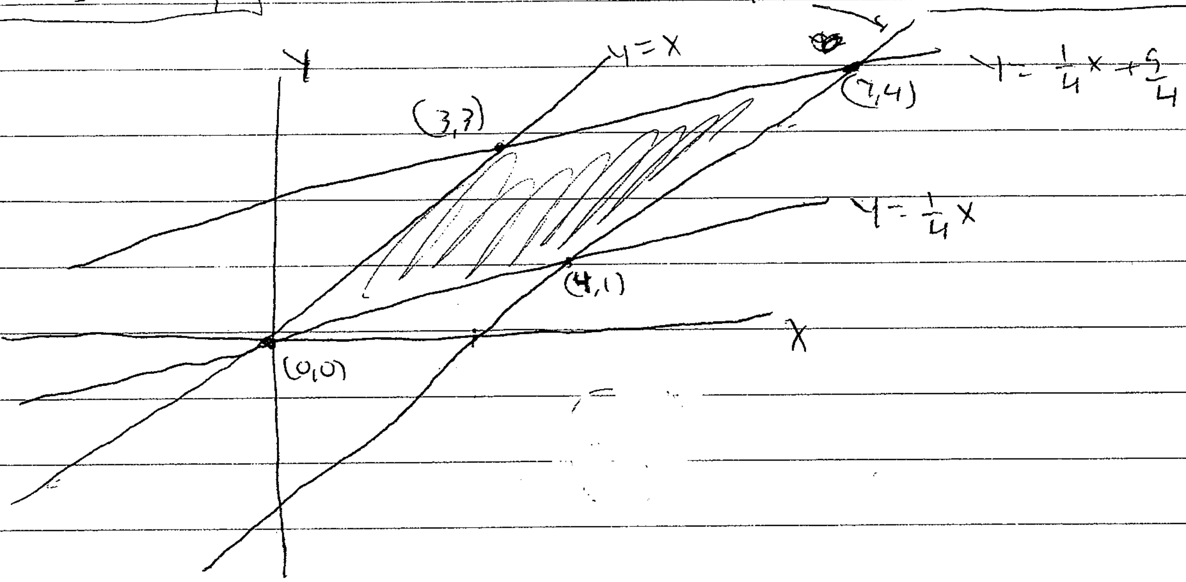
$3u = -y + x \rightarrow 12u = -4y + 4x$

$9v = 4y - x \qquad + \quad 9v = 4y - x$

$12u + 9v = 3x$

$\boxed{4u + 3v = x}$

b.

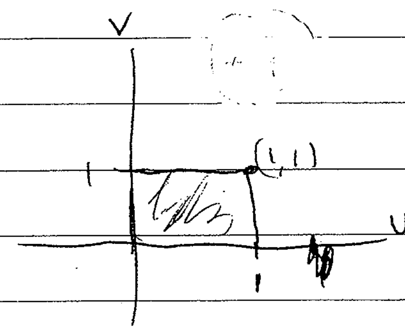


$y = x \rightarrow u + 3v = 4u + 3v \rightarrow u = 0$

$y = x - 3 \rightarrow u + 3v = 4u + 3v - 3 \rightarrow u = 1$

$y = \frac{1}{4}x \rightarrow \qquad \rightarrow v = 0$

$y = \frac{1}{4}x + \frac{9}{4} \rightarrow \qquad \rightarrow v = 1$



c. $J = \begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix} = 12 - 3 = 9$

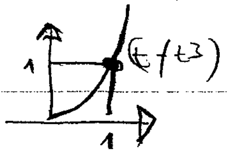
$$\int_0^1 \int_0^1 9e^{16u + 12v - u - 3v} \, du \, dv$$

$$= 9 \int_0^1 \int_0^1 e^{15u} e^{9v} \, du \, dv$$

d. $9 \int_0^1 \int_0^1 e^{15u} e^{9v} \, du \, dv = \frac{9}{15} \int_0^1 e^{15u} e^{9v} \Big|_0^1 dv = \frac{9}{15} \int_0^1 (e^{15} - 1) e^{9v} dv$

$$= \frac{1}{15}(e^{15} - 1) e^{9v} \Big|_0^1 = \boxed{\frac{1}{15}(e^{15} - 1)(e^9 - 1)}$$

$\boxed{y=x^3}$



#4 $r(t) = t\cdot i + t^3 j = \begin{pmatrix} t \\ t^3 \end{pmatrix} \in \mathbb{R}^2 \qquad 0 \le t \le 1$

in a force field $F = i + 3x^2 j = \begin{pmatrix} 1 \\ 3x^2 \end{pmatrix}$ $\qquad M = 1$, $N = 3x^2$

(a) Calculate work (flow) along C?

$$T = \frac{dr}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} \frac{dt}{dt} \\ \frac{dt^3}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix}$$

$x = t$

$$F\cdot T = \begin{pmatrix} 1 \\ 3x^2(t) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix} = 1 + 9\cdot t^4$$

$$\int_0^1 (1 + 9t^4)\, dt = \cancel{[t + }$$

$\int t^8 = \frac{1}{9} t^9$

$\int t^4 = \frac{1}{5} t^5$

$$= \left[t + \frac{9}{5} t^5\right]_0^1 = \cancel{1+1=2}\ 1 + \frac{9}{5} = 1\tfrac{9}{5} = \frac{14}{5}$$

(b) Flux $\displaystyle\int_C F\cdot n\, ds = \int_0^1 (i + 3t^2 j)\cdot(3t^2 i - j)\, dt\ |v|\, dt$

$$= \int_0^1 0 \cdot |v|\, dt = 0$$

(c) Tangent is geometrical & does not depend on the parametrization

$\begin{pmatrix} 2t \\ 2t^3 \end{pmatrix}$, $0 \le t \le \frac{1}{2}$ $\quad 0 \to 1$, $0 \to$ $\quad t^3 = \frac{1}{8}\cdot 2$

— stays the same

(d) normal is a unit vector to the curve & does not depend on the parametrization

— stays the same