INSTRUCTIONS: Electronic devices, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (30 Points) Find the absolute maximum and minimum values of the function $f(x,y) = 20 - 16x - 4y + 4x^2 + y^2$ on the closed region bounded by the lines $x = 4$, $y = x$, and $y = 0$. Be sure to clearly give both the locations and values of the absolute extremum.

2. (30 points) Consider the curve $y = \sqrt{x}$ for $0 \le x \le 3$. Find the point on the curve closest to, and furthest from, the point $(2, 0)$.

   Although this problem can be worked using simple Calculus I principles, you need to work this problem using Calculus III principles! Clearly explain your approach as you work through the problem! (Hint: you may find it useful to graph the constraint curve in the x-y plane as well as the level curves of the objective function.)

3. (30 Points) Consider a particle moving along a path in space through a temperature field $T(x, y, z)$. At a particular instant in time (and only at that instant in time) the position of the particle is $\mathbf{r} = 2\,\mathbf{i} + \mathbf{j} + \mathbf{k}$, its velocity is $\mathbf{v} = 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$, and its acceleration is $\mathbf{a} = 2\,\mathbf{j}$. At the location $(2, 1, 1)$ the value of the temperature is $T(2, 1, 1) = 5$ and the gradient of the temperature field at the same location is $\nabla T(2, 1, 1) = 3\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$.

   (a) At the position and time described, at what rate is the temperature changing with respect to time?

   (b) If the particle continues along the path C for $\Delta t = 0.1$, by how much will the temperature change?

   (c) If possible, at the location described, calculate the curvature of the path. Otherwise, clearly write "Curvature not possible."

   (d) If possible, at the location described, calculate the torsion of the path. Otherwise, clearly write "Torsion not possible."

4. (30 Points) Consider the integral $\displaystyle\iint e^{xy}\,dA$ over the region described by $10 \le xy \le 20$ and $20 \le x^2y \le 40$.

   (a) Sketch the region of integration in the xy-plane. Clearly label the boundaries of the region of integration.

   (b) To simplify the integration, try the substitution $u = xy$ and $v = x^2y$. Neatly draw the new region of integration in the u-v plane and clearly label the boundaries.

   (c) Finally, rewrite the integral in terms of u and v and evaluate it.

5. (40 Points) Consider the vector field $\mathbf{F} = 6x^2y\,\mathbf{i} + (2x^3 + 2yz)\,\mathbf{j} + y^2\,\mathbf{k}$ and the path in space C defined by $\mathbf{r}(t) = t^3\,\mathbf{i} + 2\,\mathbf{j} + t^2\,\mathbf{k}$ for $0 \le t \le 1$. (Note that the path C is in the plane $y = 2$.)

   (a) Calculate the flow of $\mathbf{F}$ along path C.

   (b) If your flow calculation in part (a) can be verified using an alternate calculation, then do so, and clearly explain your reasoning. Otherwise, clearly write "Bummer."

   (c) Calculate the outward flux across path C (in the plane $y = 2$).

   (d) If your flux calculation in part (c) can be verified using an alternate calculation, then do so, and clearly explain your reasoning. Otherwise, clearly write "Bummer."

6. (40 Points) Consider an object bounded on top by the surface $z + x^2 + y^2 = 1$ and the bottom by the surface $x^2 + y^2 + z^2 = 1$.

   (a) Find a parameterization of the curve C defined by the largest intersection of the two surfaces.

   (b) Calculate the circulation of the field given by $\mathbf{F} = 3y\,\mathbf{i} - 3x\,\mathbf{j}$ around the path C in a counterclockwise direction.

   (c) If your circulation calculation in part (b) can be verified using any theorem(s) from Calculus III, state the theorem(s), and evaluate the appropriate alternate calculation. Otherwise, clearly write "Bummer."

   (d) Calculate the outward flux of the (new) field defined by $\mathbf{G} = 2x\,\mathbf{i} + 2y\,\mathbf{j}$ across the top surface of the object.

   (e) If your net outward flux calculation in part (d) can be verified using any theorem(s) from Calculus III, state the theorem(s), and evaluate the appropriate alternate calculation. Otherwise, clearly write "Bummer."

Projections and distances $\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A}$ $\quad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$ $\quad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

The Second Derivative Test

Suppose $f(x,y)$ and its first and second partial derivatives are continuous in a disk centered at (a,b) and $f_x(a,b) = f_y(a,b) = 0$. Let $D = f_{xx}f_{yy} - f_{xy}^2$.

1. If $D > 0$ and $f_{xx} < 0$ at (a,b), then f has a local maximum at (a,b).
2. If $D > 0$ and $f_{xx} > 0$ at (a,b), then f has a local minimum at (a,b).
3. If $D < 0$ at (a,b), then f has a saddle point at (a,b).
4. If $D = 0$ at (a,b), then the test is inconclusive.

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$\begin{aligned} f(x,y) \quad = \quad & f(x_0,y_0) + \Big[(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\Big] \\ & + \frac{1}{2!}\Big[(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\Big] \\ & + \frac{1}{3!}\Big[(x-x_0)^3 f_{xxx}(x_0,y_0) + 3(x-x_0)^2(y-y_0)f_{xxy}(x_0,y_0) \\ & \qquad + 3(x-x_0)(y-y_0)^2 f_{xyy}(x_0,y_0) + (y-y_0)^3 f_{yyy}(x_0,y_0)\Big] + \cdots \end{aligned}$$

Linear approximation error

$$|E(x,y)| \le \frac{1}{2}M(|x-x_0| + |y-y_0|)^2, \qquad \text{where } \max\{|f_{xx}|,|f_{xy}|,|f_{yy}|\} \le M$$

Polar coordinates $\quad x = r\cos\theta \quad y = r\sin\theta \quad r^2 = x^2 + y^2 \quad dA = dx\,dy = r\,dr\,d\theta$

Cylindrical and spherical coordinates

Cylindrical to Rectangular	Spherical to Cylindrical	Spherical to Rectangular
$x = r\cos\theta$	$r = \rho\sin\phi$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$z = \rho\cos\phi$	$y = \rho\sin\phi\sin\theta$
$z = z$	$\theta = \theta$	$z = \rho\cos\phi$

$$dV = dx\,dy\,dz = dz\,r\,d\theta\,dr = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Substitutions in multiple integrals

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\left(x(u,v),y(u,v)\right)\,|J(u,v)|\,du\,dv \qquad \text{where} \qquad J(u,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Mass, moments, and center of mass $\quad$ Mass $\quad M = \iint_R \delta\,dA$

$$\text{Moments} \quad M_x = \iint_R y\,\delta\,dA \qquad M_y = \iint_R x\,\delta\,dA \qquad \text{Center of mass} \quad \bar{x} = M_y/M \qquad \bar{y} = M_x/M$$

Green's Theorem in a plane (The curve C is traversed counterclockwise.)

$$\begin{aligned} \text{Circulation} \quad &= \quad \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \nabla\times\mathbf{F}\cdot\mathbf{k}\,dA \\ \text{Outward Flux} \quad &= \quad \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \nabla\cdot\mathbf{F}\,dA \end{aligned}$$

Surface area of level surface $g(x,y,z) = c$ $\quad S = \iint_S d\sigma = \iint_S \frac{|\nabla g|}{|\nabla g\cdot\mathbf{p}|}\,dA$

Stoke's Theorem $\quad \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma$

Divergence Theorem of Gauss $\quad \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV$