INSTRUCTIONS: Electronic devices, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (25 points) Molly the mole has just relocated to a new plot of land and she loves the tasty little grubs that live in the ground there. To be efficient, Molly usually only tunnels around a few inches under the surface in search of these grubs. On this particular plot of land the grub density (near the surface) can be represented by the function $\rho(x, y) = 4xy - x^4 - y^4 + 100$.

   (a) Determine the location of all critical points of the grub density ρ.

   (b) Classify these critical points, and determine the value of ρ at each of these locations.

   (c) Finally, determine the x-y coordinates where Molly will find the highest concentration of grubs.

2. (25 points) Molly the mole decides to dig a tunnel through a small hill in search of more tasty grubs. Her position as a function of time is given by the position vector $\mathbf{r}(t)$. The density of grubs is given by the function $\rho(x, y, z)$. At some time t^* (and only at this time) Molly's position, velocity and acceleration are $\mathbf{r}(t^*) = 4\,\mathbf{i} + 5\,\mathbf{j} + 6\,\mathbf{k}$, $\mathbf{v}(t^*) = 1\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$, and $\mathbf{a}(t^*) = 2\,\mathbf{i} + 2\,\mathbf{j} + 6\,\mathbf{k}$ respectively. Furthermore, you know that $\nabla\rho|_{(4,5,6)} = 4\,\mathbf{i} + 0\,\mathbf{j} + 3\,\mathbf{k}$, and $\rho(4, 5, 6) = 42$.

   (a) As Molly passes location $\mathbf{r}(t^*)$, at what rate is the density ρ changing with respect to *distance*?

   (b) As Molly passes location $\mathbf{r}(t^*)$, at what rate is ρ changing with respect to *time*?

   (c) If Molly continues on her original path $\mathbf{r}(t)$ for a short interval of time $\Delta t = 0.1$, by approximately how much does the grub density change?

   (d) Now, suppose at time t^* Molly actually decides to tunnel in a direction that happens to be the direction of the greatest rate of increase of ρ. Approximately how much does the grub density change if she tunnels for a short distance $\Delta s = 0.2$?

   (e) What is the curvature of Molly's tunnel at the location $\mathbf{r}(t^*)$?

3. (25 points) Consider a rectangle of width W and height H.

   (a) Using an appropriate technique from Calculus III, show that the rectangle with maximum area A, for a given perimeter P, is actually a square.

   (b) Draw the constraint curve and several of the level curves of the objective function (the function you are trying to maximize or minimize) in the W-H plane. Hint: be sure that these curves are physically realistic.

   (c) On the graph, label all points of interest on the constraint curve. At each of these points, clearly indicate the value of the objective function and whether it is a local maximum or minimum.

4. (25 points) Consider the function $f(x,y) = x^2 + y^2 + \cos(xy)$.

(a) Calculate the *first order* Taylor approximation to $f(x,y)$ near the *origin*.

(b) Calculate an "upper bound on the error" associated with your linearization in part (a) assuming that you only use values of x and y such that $|x| \le 0.1$ and $|y| \le 0.1$.

(c) Calculate the "exact" value of $f(0.1, 0.1)$ to five decimal places. Hint: $\cos(0.01) \approx 0.99995$.

(d) Calculate the "linearization" value of $f(0.1, 0.1)$ using your result from part (a).

(e) Based on your results in part (c) and (d), calculate the "actual" error. Then, in your bluebook make a table like the one shown below and fill in the values.

$f(0.1,0.1)$ "exact"	$f(0.1,0.1)$ linearization	upper bound on error	"actual" error
XXXXX	XXXXX	XXXXX	XXXXX

Projections and distances

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$\begin{aligned} f(x,y) \;=\;& f(x_0,y_0) + \Big[(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\Big] \\ &+\frac{1}{2!}\Big[(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\Big] \\ &+\frac{1}{3!}\Big[(x-x_0)^3 f_{xxx}(x_0,y_0) + 3(x-x_0)^2(y-y_0)f_{xxy}(x_0,y_0) \\ &\qquad + 3(x-x_0)(y-y_0)^2 f_{xyy}(x_0,y_0) + (y-y_0)^3 f_{yyy}(x_0,y_0)\Big] + \cdots \end{aligned}$$

Linear approximation error

$$|E(x,y)| \le \frac{1}{2}M(|x-x_0| + |y-y_0|)^2, \qquad \text{where } \max\{\,|f_{xx}|,\,|f_{xy}|,\,|f_{yy}|\,\} \le M$$