

INSTRUCTIONS: Books, notes, crib sheets, and electronic devices are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show and explain your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (25 points) You have discovered how to make glow-in-the-dark tattoo ink. Your “Glottoos” are a huge success since nobody can see them till you're home and it's dark. Unfortunately, the ink is extremely expensive and each section of a tattoo needs to have its area calculated very accurately so you know how much to ~~buy~~ charge the customer. Having done well in your Calculus III course, you determine that the area covered for a particular section of a tattoo is given by

$$A = \int_{y=0}^1 \int_{x=y^2}^{\sqrt[3]{y}} dx dy .$$

Your co-worker (a more recent Calculus III wunderstudent) suggests that it might be fun to change the order of integration. And so you set to work...

- Sketch the shape/region of the tattoo section and clearly indicate the axis, boundary equations, intersection points, etc.
 - Rewrite the integral by switching the order of integration.
 - Evaluate the area of the tattoo section using your results from part (b).
2. (25 points) Later, you discover that by varying the density of ink you can produce a 3-D effect in a Glottoo because some parts glow brighter than others. For a particular customer, after accounting for the shape of the Glottoo and the desired ink density, you have calculated that the total cost, C , can be determined from the integral

$$C = \iint_{R_{xy}} e^{(x+y)/(x-y)} dx dy ,$$

where R_{xy} represents the Glottoo region in an xy -plane bounded by the curves $y = 0$, $y = x - 2$, $x = 0$, and $y = x - 1$. Your co-worker suggests trying a variable substitution to determine the value of C . Again, you set to work...

- You realize that the substitution $u = x + y$ and $v = x - y$ greatly simplifies the evaluation of C . Find x and y in terms of u and v using the given substitution. Be sure to check this because the rest of the problem depends on this result!
 - To evaluate C in terms of the new variables, you need to transform the original region R_{xy} into its corresponding region R_{uv} in the uv -plane. Make two clear sketches, one of the original region of integration R_{xy} in the xy -plane, and one of the new region of integration R_{uv} in the uv -plane. Be sure to label all axes, boundaries, intersection points, etc. on each sketch.
 - Rewrite the integral for C over the region R_{uv} in the uv -plane in terms of u and v .
 - Evaluate C in terms of u and v .
3. (25 points) The integrals

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1/\sqrt{2}} \int_{z=0}^r r dz dr d\theta + \int_{\theta=0}^{2\pi} \int_{z=0}^{1/\sqrt{2}} \int_{r=1/\sqrt{2}}^{\sqrt{1-z^2}} r dr dz d\theta$$

describe the volume of an object. Warning: take note of the order of integration above!

- Make a clear sketch of the region of integration in the xyz -coordinate system clearly labeling the bounding surfaces of the region of integration. (If you have trouble with this, you may “buy” a sketch of the shape of the region of integration for 5 points. **This sketch will only show the shape of the region**, so you will still need to supply the remaining details.)
- Express V as a single integral in cylindrical coordinates using the order $dr dz d\theta$.
- Express V in spherical coordinates using the order $d\theta d\rho d\phi$.
- Express V in spherical coordinates using the order $d\rho d\phi d\theta$.
- Evaluate one of the integrals above to determine the value of V .

OVER

4. (25 points) Consider an object moving along a path C in the xy -plane given by $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}$ from the point $(0, 0)$ to $(1, 1)$ in a force field given by $\mathbf{F} = x\mathbf{i} + xy\mathbf{j}$.

(a) Using $\mathbf{r}_1(t)$, calculate the **flow** along the path C .

(b) Using $\mathbf{r}_1(t)$, calculate the **flux** across the path C .

(c) Determine another reasonable parametrization, $\mathbf{r}_2(t)$, for path C and verify your result in part (a) using $\mathbf{r}_2(t)$. Be sure that your $\mathbf{r}_2(t)$ actually takes you along the same path as $\mathbf{r}_1(t)$!

Projections and distances $\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}}\right)\mathbf{A}$ $d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$ $d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}| dt \quad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \quad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \quad \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|1 + (f'(x))^2|^{3/2}} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{|\dot{x}^2 + \dot{y}^2|^{3/2}} \quad \tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \quad a_T = \frac{d|\mathbf{v}|}{dt} \quad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f) \cdot \mathbf{u} \quad f_{xx}f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda\nabla g, \quad g = 0$$

Polar coordinates $x = r \cos \theta$ $y = r \sin \theta$ $r^2 = x^2 + y^2$ $dA = dx dy = r dr d\theta$

Cylindrical and spherical coordinates

Cylindrical to Rectangular	Spherical to Cylindrical	Spherical to Rectangular
$x = r \cos \theta$	$r = \rho \sin \phi$	$x = \rho \sin \phi \cos \theta$
$y = r \sin \theta$	$z = \rho \cos \phi$	$y = \rho \sin \phi \sin \theta$
$z = z$	$\theta = \theta$	$z = \rho \cos \phi$

$$dV = dx dy dz = r dr d\theta dz = \rho^2 \sin \phi d\rho d\phi d\theta$$

Substitutions in multiple integrals

$$\iint_R f(x, y) dx dy = \iint_G f(x(u, v), y(u, v)) |J(u, v)| du dv \quad \text{where} \quad J(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

Mass, moments, and center of mass Mass $M = \iint_R \delta dA$

Moments $M_x = \iint_R y \delta dA$ $M_y = \iint_R x \delta dA$ Center of mass $\bar{x} = M_y/M$ $\bar{y} = M_x/M$

Flow and flux Flow = $\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot \mathbf{V} dt = \int_C \mathbf{F} \cdot d\mathbf{r}$ Flux = $\int_C \mathbf{F} \cdot \mathbf{n} ds$