Name: ____________________

APPM 2350 Exam #2 Summer 2008

Be sure to include your name and a grading table on the front of your blue book. You must work all of the problems on this exam. Show ALL of your work and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, a wrong answer with no work will receive no credit, and an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, crib sheets, cell phones, calculators, or electronic devices of any kind are NOT permitted. Please clearly indicate the start of each new problem. Good luck!

1. (20 points) Consider $f(x,y) = \dfrac{x^2 - y^2}{xy}$

   (a) Where is $f(x,y)$ continuous?

   $f(x,y)$ is continuous for all points except $xy = 0$ since this is undefined. Therefore the lines $x = 0$ and $y = 0$ (the y-axis and x-axis, respectively) are places where $f(x,y)$ is not continuous.

   (b) What is $\lim_{(x,y)\to(0,0)} f(x,y)$?

   Consider approaching the origin along $y = kx$ for $k \in \mathbb{R}$ and $k \neq 0$. Then $f(x,y)$ takes the following form

   $$f(x,kx) = \frac{x^2 - (kx)^2}{x(kx)} = \frac{x^2(1-k^2)}{x^2 k} = \frac{(1-k^2)}{k}$$

   So $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,kx) = \dfrac{(1-k^2)}{k}$ which depends on the slope k of the line approaching the origin. Since the limit varies with the approach, the limit does not exist.

   (c) Find all first partial derivatives of $f(x,y)$. Simplifying your answers will be helpful.

   $$\begin{aligned} f_x(x,y) &= \frac{2x^2y - y(x^2-y^2)}{(xy)^2} = \frac{x^2y + y^3}{(xy)^2} \\ f_y(x,y) &= \frac{-2xy^2 - x(x^2-y^2)}{(xy)^2} = \frac{-xy^2 - x^3}{(xy)^2} \end{aligned}$$

   (d) If $x(t) = \sin(t)$ and $y(t) = \cos(t)$, find $\frac{df}{dt}$.

   Chain rule yields

   $$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

Observe $\frac{dx}{dt} = \cos(t) = y$ and $\frac{dy}{dt} = -\sin(t) = -x$ so

$$\begin{aligned}
\frac{df}{dt} &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \\
&= \frac{x^2 y + y^3}{(xy)^2} y + \frac{-xy^2 - x^3}{(xy)^2}(-x) \\
&= \frac{x^2y^2 + y^4}{(xy)^2} + \frac{x^2y^2 + x^4}{(xy)^2} \\
&= \frac{x^4 + 2x^2y^2 + y^4}{(xy)^2} \\
&= \frac{(x^2+y^2)^2}{(xy)^2}
\end{aligned}$$

Substituting $x = \sin(t)$ and $y = \cos(t)$ into the last expression gives

$$\begin{aligned}
\frac{df}{dt} &= \frac{(1)^2}{(\sin(t)\cos(t))^2} = \frac{1}{\left(\frac{\sin(2t)}{2}\right)^2} \\
&= 4\csc^2(2t)
\end{aligned}$$

2. (20 points) Let $f(x, y) = \sin(x - y)$.

(a) Determine the linearization of $f(x, y)$ at the origin.

Observe

$f_x = \cos(x-y)$	$f_y = -\cos(x-y)$	
$f_{xx} = -\sin(x-y)$	$f_{yy} = -\sin(x-y)$	$f_{xy} = \sin(x-y)$

So the linearization is

$$\begin{aligned}
L(x, y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\
&= x - y
\end{aligned}$$

(b) Bound the error in this approximation if $-\frac{1}{10} \leq x \leq \frac{1}{10}$ and $-\frac{1}{10} \leq y \leq \frac{1}{10}$.

Note that $|f_{xx}| = |-\sin(x-y)| \leq 1$, $|f_{xy}| = |\sin(x-y)| \leq 1$ and $|f_{yy}| = |-\sin(x-y)| \leq 1$ for all $x, y \in \mathbb{R}^2$. So

$$|E(x,y)| \leq \frac{1}{2}(|x| + |y|)^2 \leq \frac{1}{2}\left(\tfrac{1}{10} + \tfrac{1}{10}\right)^2 = \frac{1}{50}$$

(c) What is the quadratic approximation to $f(x, y)$ at the origin?

The quadratic approximation to $f(x,y)$ is given by

$$\begin{aligned} f(x,y) &\doteq f(0,0) + f_x(0,0)x + f_y(0,0)y \\ &\quad + \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right) \\ &= x - y \end{aligned}$$

since all second order partial derivatives evaluate to zero at $(x,y) = (0,0)$.

3. (20 points) A heated storage room is shaped like a rectangular box and has a volume of 1000 cubic meters. Because warm air rises, the heat loss per unit area through the ceiling is five times as great as the heat loss through the floor. If the heat loss through the four walls is three times as great as the heat loss through the floor, determine the room dimensions that will minimize heat loss and thus minimize the heating costs. **In order to receive full credit** for this question, you must demonstrate the solution minimizes the heat loss by evaluating the objective function at another point on the constraint-surface.

   The constraint is

   $$V(x,y,z) = xyz = 1000$$

   Assuming that the area of the floor is xy, and the area of one end is xz, then the objective function is

   $$\begin{aligned} h(x,y,z) &= xy + 3(2)xz + 3(2)yz + 5xy \\ &= 6xy + 6xz + 6yz \end{aligned}$$

   So $\nabla V = \lambda \nabla h$ is equivalent to the system of equations

   $$\begin{aligned} yz &= \lambda(6y + 6z) & (1) \\ xz &= \lambda(6x + 6z) & (2) \\ xy &= \lambda(6x + 6y) & (3) \end{aligned}$$

   Multiply (1) by x, (2) by y, and (3) by z to obtain

   $$\begin{aligned} xyz &= \lambda(6xy + 6xz) & (4) \\ xyz &= \lambda(6xy + 6yz) & (5) \\ xyz &= \lambda(6xz + 6yz) & (6) \end{aligned}$$

   Observe, if $x = 0$, $y = 0$, or $z = 0$ then we have a contradiction with the constraint, so $x \neq 0$, $y \neq 0$, and $z \neq 0$. Subtracting equation (5) from (4) gives

   $$0 = \lambda(6xz - 6yz) = 6\lambda z(x - y)$$

   Since $z \neq 0$, it follows $x = y$. Similarly, subtract equation (6) from (4) giving

   $$0 = \lambda(6xy - 6yz) = 6\lambda y(x - z)$$

Since $y \neq 0$, it follows $x = z$. So we have $x = y = z$ and substituting this relationship into the constraint implies $x^3 = 1000$ so $x = y = z = 10$. Now, consider another point $P'(x', y', z') = (250, 2, 2)$ satisfying the constraint $V = 1000$. Evaluating $h(x, y, z)$ at P gives 6024. Since $h(10, 10, 10) < h(250, 2, 2)$, we have indication that indeed the result is a minimum.

4. (20 points) Let

$$f(x, y, z) = \frac{(x+1)^2(y+1)^2}{2} + e^{xy}z^2$$

be a function defined on $\mathbb{R}^3$.

(a) Compute the gradient of $f(x, y, z)$.

$$\begin{aligned} \nabla f &= \left[(x+1)(y+1)^2 + ye^{xy}z^2\right]\hat{\mathbf{i}} \\ &\quad + \left[(x+1)^2(y+1) + xe^{xy}z^2\right]\hat{\mathbf{j}} \\ &\quad + 2e^{xy}z\hat{\mathbf{k}} \end{aligned}$$

(b) Compute the directional derivative of $f(x, y, z)$ in the direction of the vector $\mathbf{w} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ at the point $P(0, 0, 1)$.

We can see $\nabla f|_P = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. In addition, let $\hat{\mathbf{w}} = \frac{\mathbf{w}}{|\mathbf{w}|} = \frac{1}{3}\left(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}\right)$.

$$\begin{aligned} D_{\hat{\mathbf{w}}} &= \nabla f|_P \cdot \hat{\mathbf{w}} \\ &= -\tfrac{1}{3} \end{aligned}$$

For parts (c) and (d), also consider the unit sphere centered at the origin and described as the level surface

$$g(x, y, z) = x^2 + y^2 + z^2 = 1$$

(c) If a **unit vector**

$$\hat{\mathbf{u}} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$$

exists in the plane tangent to the unit sphere at the point $P(0, 0, 1)$, what has to be true about the relationship between $\hat{\mathbf{u}}$ and $\nabla g|_P$?

The described situation requires that $\nabla g|_P \cdot \hat{\mathbf{u}} = 0$. In other words, every vector in the tangent plane is orthogonal to the normal vector at P.

(d) Is there a unit vector in the plane tangent to the sphere at the point $P(0, 0, 1)$ such that the directional derivative $D_{\hat{\mathbf{u}}}f$ is equal to $\frac{1}{2}$?

We seek a vector $\hat{\mathbf{u}}$ such that $|\hat{\mathbf{u}}|^2 = u_1^2 + u_2^2 + u_3^2 = 1$. In addition, from part (c), we know $\nabla g|_P \cdot \hat{\mathbf{u}} = 0$. Since $\nabla g = \langle 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}\rangle$, this relationship can be expressed as

$$\begin{aligned} \nabla g|_P \cdot \hat{\mathbf{u}} &= 2\hat{\mathbf{k}} \cdot (u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}) = 0 \\ \Rightarrow u_3 &= 0 \end{aligned}$$

Therefore our goal is to find a $\hat{\mathbf{u}}$, such that

$$\begin{aligned} |\hat{\mathbf{u}}| &= u_1^2 + u_2^2 + u_3^2 = 1 \\ u_3 &= 0 \\ D_{\hat{\mathbf{u}}} f &= \nabla f_P \cdot \hat{\mathbf{u}} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \cdot (u_1 \hat{\mathbf{i}} + u_2 \hat{\mathbf{j}} + u_3 \hat{\mathbf{k}}) = \tfrac{1}{2} \end{aligned}$$

Putting this together implies

$$\begin{aligned} |\hat{\mathbf{u}}| &= u_1^2 + u_2^2 = 1 \\ u_3 &= 0 \\ u_1 + u_2 &= \tfrac{1}{2} \end{aligned}$$

Solving the third equality for u_2 and substituting into the first equality gives

$$2u_1^2 - u_1 - \tfrac{3}{4} = 0$$

Using the quadratic formula we see $u_1 = \frac{1 \pm \sqrt{7}}{4}$. So there are two directions that live in the tangent plane where the directional derivative of f in that direction is equal to $\frac{1}{2}$. They are

$$\hat{\mathbf{u}} = \tfrac{1+\sqrt{7}}{4}\hat{\mathbf{i}} + \tfrac{1-\sqrt{7}}{4}\hat{\mathbf{j}} \quad \text{and } \hat{\mathbf{u}} = \tfrac{1-\sqrt{7}}{4}\hat{\mathbf{i}} + \tfrac{1+\sqrt{7}}{4}\hat{\mathbf{j}}$$

5. (20 points) Let the electric potential, $P(x, y)$, on a circular region be given by

$$P(x, y) = 100 - 2x^2 - y^2$$

At $t = 0$, assume a negative test charge is placed inside this region at the point $(3, 3)$. At each instance, this test charge will move in a direction so that the increase in electric potential is a maximum. In addition, it moves with a **speed** equal to the absolute value of the change in electric potential at its instantaneous position.

(a) In what **direction** does the charge initially move and with what **speed**?

The thing to realize is the charge has velocity $\mathbf{v}(t) = \nabla P(x(t), y(t))$. Then

$$\nabla P(x, y) = -4x\hat{\mathbf{i}} - 2y\hat{\mathbf{j}} \Rightarrow \nabla P(3, 3) = -12\hat{\mathbf{i}} - 6\hat{\mathbf{j}}$$

So the direction is given by $\frac{\nabla P(3,3)}{|\nabla P(3,3)|} = \frac{1}{\sqrt{180}}(-12\hat{\mathbf{i}} - 6\hat{\mathbf{j}}) = \frac{1}{6\sqrt{5}}(-12\hat{\mathbf{i}} - 6\hat{\mathbf{j}}) = \frac{1}{\sqrt{5}}(-2\hat{\mathbf{i}} - \hat{\mathbf{j}})$

(b) Draw several level curves of $P(x, y)$ in the xy-plane. In addition, sketch the path of the charge.

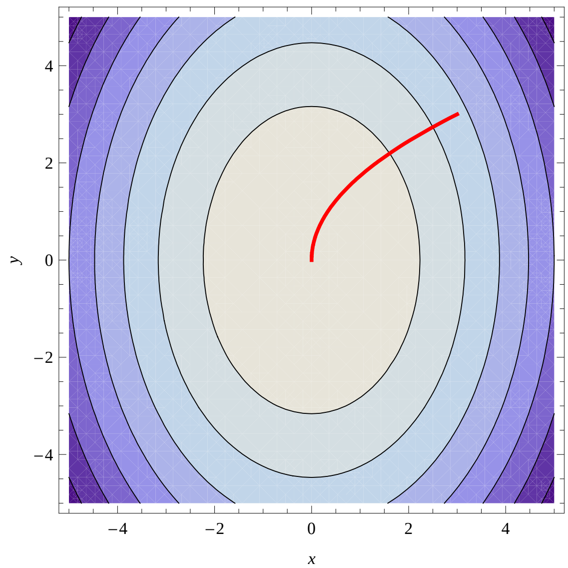

Figure 1: The path of the particle when started from $(3, 3)$ indicated by red curve.

(c) If the velocity of the charge is expressed as $\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}}$, find the parameterization of $\mathbf{r}(t)$.

Assuming $\mathbf{v}(t)$ has the expression above and using the fact that $\mathbf{v}(t) = \nabla P(x, y)$, we have

$$\begin{aligned} \frac{dx}{dt} &= -4x \\ \frac{dy}{dt} &= -2y \end{aligned}$$

Consider just $\frac{dx}{dt} = -4x$. Then

$$\begin{aligned} \frac{dx}{dt} &= -4x \\ \iff \frac{dx}{x} &= -4tdt \\ \iff \int \frac{dx}{x} &= \int -4dt \\ \iff \ln(x) &= -4t + c \quad \text{where } c \in \mathbb{R} \\ \iff x &= Ce^{-4t} \quad \text{where } C \in \mathbb{R} \end{aligned}$$

Using the initial condition $(x(0), y(0)) = (3, 3)$, then $x(0) = Ce^0 = C = 3$, so $x(t) = 3e^{-4t}$. Using the same argument on $\frac{dy}{dt}$, we can obtain $y(t) = 3e^{-2t}$.