Calc 3 Ex 3 Sum '08 versions A+B

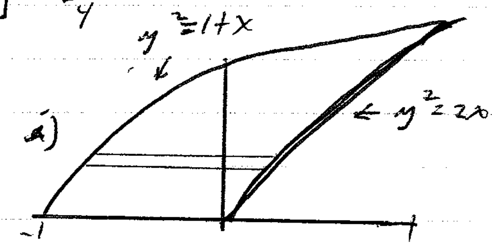
1) a) $$\int_{\pi/6}^{\pi/2}\int_0^1 \frac{(1-r^2)\cos\theta}{\sin\theta}\, r\,dr\,d\theta$$

Same on Both

$$\int_{\pi/6}^{\pi/2} \frac{\cos\theta}{\sin\theta}\left[\frac{r^2}{2}-\frac{r^4}{4}\Big|_0^1\right]d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/2}\frac{\cos\theta\,d\theta}{\sin\theta}$$

$u = \sin\theta$, $du = \cos\theta$, $\theta = \pi/6$ $u = 1/2$, $\theta = \pi/2$ $u = 1$

$$\frac{1}{4}\int_{1/2}^{1}\frac{du}{u} = \frac{1}{4}\ln u\Big|_{1/2}^{1} = \boxed{-\frac{1}{4}\ln \frac{1}{2}} = \frac{\ln 2}{4}$$




b) $$\int_{-1}^{0}\int_{0}^{\sqrt{1+x}} dy\,dx + \int_0^1\int_{\sqrt{2x}}^{\sqrt{1+x}} dy\,dx$$

$y=\sqrt{1+x} \Rightarrow y^2 = 1+x$

$y = \sqrt{2x} \Rightarrow y^2 = 2x$

$1+x = 2x \Rightarrow x=1$, $y=\sqrt{2}$

ii) $$\int_0^{\sqrt{2}}\int_{y^2-1}^{\frac{y^2}{2}} dx\,dy$$

iii) $$\int_0^{\sqrt{2}}\left[x\Big|_{y^2-1}^{y^2/2}\right]dy = \int_0^{\sqrt{2}}\left[\frac{y^2}{2}-\left(y^2-1\right)\right]dy = \int_0^{\sqrt{2}} 1-\frac{y^2}{2}\,dy$$

$$y - \frac{y^3}{6}\Big|_0^{\sqrt{2}} = \sqrt{2} - \frac{2\sqrt{2}}{6} = \sqrt{2} - \frac{\sqrt{2}}{3} = \sqrt{2}\left(\frac{2}{3}\right)$$

c) $$\iiint_R \sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi/3}\int_0^2 \rho\,(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

$$\frac{\rho^4}{4}\Big|_0^2 = 4 \qquad 4\int_0^{2\pi}\int_0^{\pi/3}\sin\phi\,d\phi\,d\theta = -4\int_0^{2\pi}\cos\phi\Big|_0^{\pi/3}d\theta$$

$$-4\int_0^{2\pi}\left(\frac{1}{2}-1\right)d\theta = 2\int_0^{2\pi}d\theta = 2\,\theta\Big|_0^{2\pi} = 4\pi$$

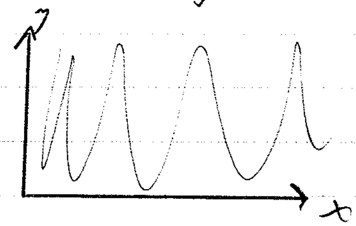
d) $$\int_0^2\int_0^{\sqrt{2-x^2}}\int_0^{\sqrt{x}} yz\,dy\,dz\,dx = \int_0^2\int_0^{\sqrt{2-x^2}} z\left[\frac{y^2}{2}\Big|_0^{\sqrt{x}}\right]dz\,dx$$

$$\int_0^2\int_0^{\sqrt{2-x^2}}\frac{zx}{2}\,dz\,dx = \int_0^2\left[\frac{z^2x}{4}\Big|_0^{\sqrt{2-x^2}}\right]dx = \frac{1}{4}\int_0^2 2x-x^3\,dx = \frac{1}{4}\left[x^2-\frac{x^4}{4}\Big|_0^2\right] = 0$$

2) $I^2 = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

#5 in version B

let $r^2 = x^2+y^2$




$$\int_0^{\pi/2}\int_0^\infty r e^{-r^2}\,dr\,d\theta$$

$u=-r^2$ $\quad du=-2r\,dr$ $\quad$ as $r\to\infty$ $\quad u\to-\infty$

$$-\frac{1}{2}\int_0^{\pi/2}\int_0^{-\infty} e^u\,du\,d\theta = -\frac{1}{2}\int_0^{\pi/2}\left[e^u\Big|_0^{-\infty}\right]d\theta = -\frac{1}{2}\int_0^{\pi/2}(-1)\,d\theta$$

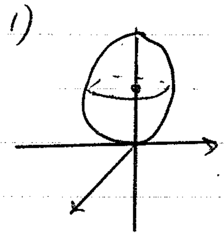
$$= \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{1}{2}\theta\Big|_0^{\pi/2} = \frac{\pi}{4} = I^2 \qquad I = \frac{\sqrt{\pi}}{2}$$

3) #2 in VB

a) $x^2+y^2+z^2=2z$

$x^2+y^2+z^2-2z+1=1$ $\qquad x^2+y^2+(z-1)^2=1$

sphere of radius $=1$ centered at $(0,0,1)$



b) $\phi = \sin\pi$ $\qquad \phi=0$

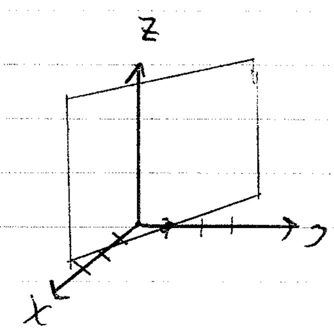
positive z axis

since $\rho\geq 0$

c) $\rho = \frac{\csc\phi}{\cos\theta+\sin\theta}$ $\qquad \rho(\cos\theta+\sin\theta) = \frac{1}{\sin\phi}$

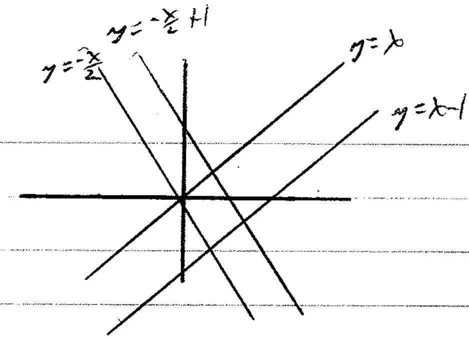
$\rho\sin\phi\,(\cos\theta+\sin\theta)=1$

$x+y=1$ plane



3 in Version B

4) a) $\iint \frac{x+2y}{\sqrt{x-y}}$




$y=x \qquad y=-\frac{x}{2}$

$y=x-1 \qquad y=-\frac{x}{2}+1$

b) $u=x+2y \qquad u+2v = x+2y+(2x-2y) = 3x$

$\ominus \; v = x-y$

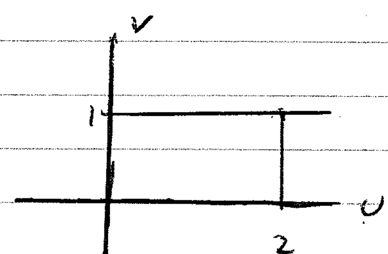
$u-v=3y \qquad y=\frac{u-v}{3} \qquad x=\frac{u+2v}{3}$

$y=x \qquad \frac{u-v}{3} = \frac{u+2v}{3} \qquad u-v=u+2v \qquad 3v=0 \qquad v=0$

$y=x-1 \qquad \frac{u-v}{3} = \frac{u+2v}{3}-1 \qquad u-v=u+2v-3 \qquad 3v=3 \qquad v=1$

$y=-\frac{x}{2} \qquad \frac{u-v}{3} = -\frac{1}{2}\left(\frac{u+2v}{3}\right) \qquad 2u-2v=-u-2v \qquad 3u=0 \qquad u=0$

$y=-\frac{x}{2}+1 \qquad \frac{u-v}{3} = -\frac{1}{2}\left(\frac{u+2v}{3}\right)+1 \qquad 2u-\cancel{2v} = -u-\cancel{2v}+6 \qquad 3u=6 \qquad u=2$




c) $x_u = \frac{1}{3} \qquad x_v = \frac{2}{3}$

$y_u = \frac{1}{3} \qquad y_v = -\frac{1}{3}$

$$\begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

d) $\int_0^2 \int_0^1 \frac{u}{\sqrt{v}} \left|\frac{-1}{3}\right| dv\,du \qquad \frac{2}{3}\int_0^2 u \left[ v^{1/2} \Big|_0^1 \right] du = \frac{2}{3}\int_0^2 u\,du$

$\frac{\cancel{2}}{3} \frac{u^2}{\cancel{2}} \Big|_0^2 \qquad \frac{4}{3}$

5) $\iint_R \frac{50}{(1+x^2+y^2)^2}\, dA$

4 in
VB

$$\int_0^{2\pi}\int_0^1 \frac{50}{(1+r^2)^2}\, r\, dr\, d\theta \qquad u = 1+r^2 \quad r=0 \quad u=1$$
$$du = 2r\,dr \quad r=1 \quad u=2$$

$$\frac{50}{2}\int_0^{2\pi}\int_1^2 \frac{1}{u^2}\, du\, d\theta \Rightarrow 25\int_0^{2\pi}\left[\frac{-1}{u}\Big|_1^2\right] d\theta$$

$$25\int_0^{2\pi}\left[-\frac{1}{2} - -\frac{1}{1}\right] d\theta = 25\int_0^{2\pi} \frac{1}{2}\, d\theta = \frac{25}{2}\left[\theta\Big|_0^{2\pi}\right] = 25\pi$$

~~25π < 50 people answer yes.~~

$25\pi < 100$ people so NO.