Final Exam Sol'ns

1.) $\underline{F} = \langle 20x^3z + 2y^2,\ 4xy,\ 5x^4 + 3z^2 \rangle$

$$\Rightarrow \nabla \times \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 20x^3z + 2y^2 & 4xy & 5x^4 + 3z^2 \end{vmatrix}$$

$$= 0\hat{i} - (20x^3 - 20x^3)\hat{j} + (4y - 4y)\hat{k} = \underline{0}$$

$\Rightarrow \int_C \underline{F} \cdot d\underline{r}$ is path independent...

Need $\underline{F} = \nabla f$

so $\hat{i}$: $\frac{\partial f}{\partial x} = 20x^3z + 2y^2 \Rightarrow f = 5x^4z + 2xy^2 + g(y,z)$

taking ∂_y of

$\hat{j}$: $\frac{\partial f}{\partial y} = 4xy + \frac{\partial g}{\partial y} = 4xy \Rightarrow \frac{\partial g}{\partial y} = 0$ so $g = g(z)$

using $\underline{F}$

$\Rightarrow f = 5x^4z + 2xy^2 + g(z)$

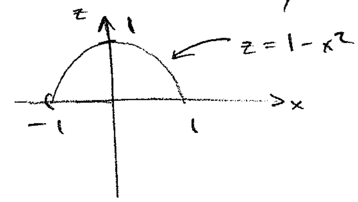
$\hat{k}$: $\frac{\partial f}{\partial z} = 5x^4 + g'(z) = 5x^4 + 3z^2 \Rightarrow g'(z) = 3z^2$ so $g(z) = z^3 + k$

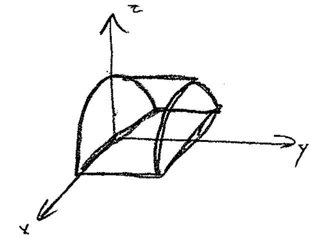
$\Rightarrow f(x,y,z) = 5x^4z + 2xy^2 + z^3 + k$

$$\Rightarrow \text{flow} = \int_{t=0}^{t=\pi} \underline{F} \cdot \frac{d\underline{r}}{dt}\,dt = f(\underline{r}(\pi)) - f(\underline{r}(0))$$
$$= f(-2, 2\pi, 0) - f(2,0,0)$$
$$= (0 + 2(-2)(2\pi)^2 + 0 + k)$$
$$- (0 + 0 + 0 + k) = -16\pi^2$$

2.) $\underline{F} = (x+\cos y)\hat{i} + (y+\sin z)\hat{j} + (z+e^{x})\hat{k}$

surface bndd by $z=0$, $y=0$, $y=2$, $z=1-x^2$





$$\text{Flux} = \iint_S \underline{F}\cdot\hat{n}\,d\sigma = \iiint_D \nabla\cdot\underline{F}\,dV$$

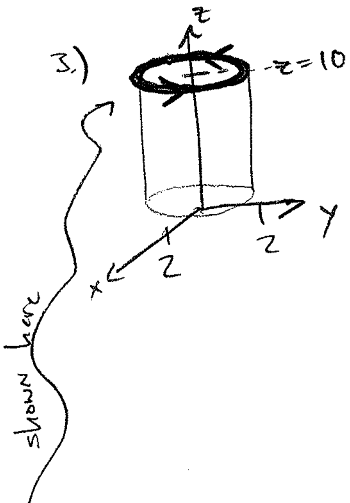
$$\nabla\cdot\underline{F} = \partial_x(x+\cos y) + \partial_y(y+\sin z) + \partial_z(z+e^{x}) = 1+1+1 = 3$$

$$\Rightarrow \text{Flux} = \iiint_D 3\,dV = 3\int_0^2\int_{-1}^1\int_0^{1-x^2} dz\,dx\,dy$$

$$= 3\left(\int_0^2 dy\right)\left(\int_{-1}^1\int_0^{1-x^2} dz\,dx\right) = 3(2)\left(\int_{-1}^1 (1-x^2)\,dx\right) = 6\left[x - \frac{x^3}{3}\right]_{-1}^1$$

$$= 6\left(\left(1-\tfrac{1}{3}\right) - \left(-1+\tfrac{1}{3}\right)\right) = 6\left(2-\tfrac{2}{3}\right) = (6)\tfrac{4}{3} = \boxed{8}$$

3.)



$$\underline{F} = -y\sqrt{x^2+y^2}\,\hat{i} + x\sqrt{x^2+y^2}\,\hat{j}$$

Circulation around upper surface of cylindrical container is

$$\iint_S (\nabla\times\underline{F})\cdot\hat{n}\,d\sigma \underset{\text{By Stokes}}{=} \oint_C \underline{F}\cdot d\underline{r}$$

Let $\underline{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 10\,\hat{k}$ for $t\in[0,2\pi]$ parametrize intersection of cylinder $x^2+y^2=2^2$ w/ plane $z=10$.

$$\Rightarrow \frac{d\underline{r}}{dt} = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} \quad \Rightarrow \quad \underline{F}\cdot\frac{d\underline{r}}{dt} = \underbrace{y^2\sqrt{x^2+y^2} + x^2\sqrt{x^2+y^2}}$$

$$= -y(t)\,\hat{i} + x(t)\,\hat{j}$$

where dependence on t has been suppressed.

$$= \sqrt{x^2+y^2}\,(x^2+y^2) = (x^2+y^2)^{3/2}$$

$$\Rightarrow \oint_C \underline{F}\cdot\frac{d\underline{r}}{dt}\,dt = \int_0^{2\pi} (x^2+y^2)^{3/2}\,dt = \int_0^{2\pi}\left[(2\cos t)^2 + (2\sin t)^2\right]^{3/2} dt = \int_0^{2\pi} 8\,dt$$

$$\boxed{=16\pi}$$

4.) a.) $\underline{v} = \hat{\imath} + 3\hat{\jmath} - \hat{k}$

$\underline{w} = -\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

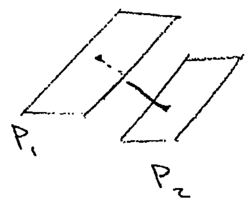
$$\text{proj}_{\underline{w}}\underline{v} = \frac{\underline{w}\cdot\underline{v}}{(\underline{w}\cdot\underline{w})}\,\underline{w} = \boxed{\frac{7}{9}\,\underline{w}}$$

$\underline{w}\cdot\underline{v} = -1 + 6 + 2$

$= 7$

$\underline{w}\cdot\underline{w} = 1 + 4 + 4 = 9$

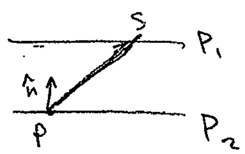
b.)



choose a pnt on $P_1 : 3x - y - 2z = 6$

so $S(2,0,0)$ is on P_1

distance between this pnt & P_2 is



$$d = \left| \overline{PS} \cdot \frac{\hat{n}}{|\underline{n}|} \right|$$

where $\underline{n} = 3\hat{\imath} - \hat{\jmath} - 2\hat{k}$

$\Rightarrow |\underline{n}| = \sqrt{9+1+4} = \sqrt{14}$

$$\Rightarrow \quad d = \frac{1}{\sqrt{15}}\,|\overline{PS}\cdot\underline{n}|$$

$\overline{PS} = \langle 0,2,0\rangle - \langle 2,0,0\rangle = -2\hat{\imath} + 2\hat{\jmath} \quad \Rightarrow \quad \underline{PS}\cdot\underline{n} = \langle -2,2,0\rangle \cdot \langle 3,-1,-2\rangle$

$= -6 - 2 = -8$

$$\Rightarrow \boxed{d = \frac{8}{\sqrt{14}}}$$

5.) a.) $f(x,y) = 2x^2 + 6y^2 \implies f_x = 4x \qquad f_y = 12y$

Tangent plane gvn by 1^{st} order Taylor approx.

$$\implies T(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

@ $(x_0,y_0) = (1,-1)$

$f(x_0,y_0) = 8$

$f_x(x_0,y_0) = 4$

$f_y(x_0,y_0) = -12 \implies T(x,y) = 8 + 4(x-1) - 12(y+1)$

$z = 8 + 4x - 4 - 12y - 12$

let $z = T(x,y)$ then tangent plane given by

$$\boxed{4x - 12y - z = 8}$$

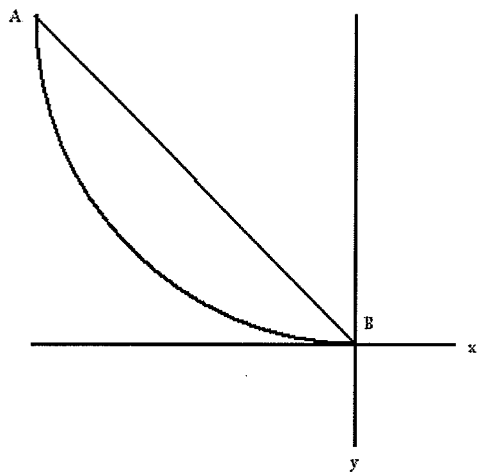
b.) Normal vector to plane is $\underline{v} = 4\hat{\imath} - 12\hat{\jmath} - \hat{k}$

$$\implies \left.\begin{aligned} x(t) &= 4t + 1 \\ y(t) &= -12t - 1 \\ z(t) &= -t + 8 \end{aligned}\right\} \text{ params line Normal to plane through } (1,-1,8).$$

6. (15 points) Two paths connect point $A(-1,1)$ to point $B(0,0)$ as in the attached picture.

   (a) Path 1 is a straight line. Write a parameterization for that line.

   (b) Path 2 is a quarter circle. Write a parameterization for the quarter circle.



a) $r(t) = (t-1)\hat{\imath} + (1-t)\hat{\jmath}$ for $t \in [0,1]$

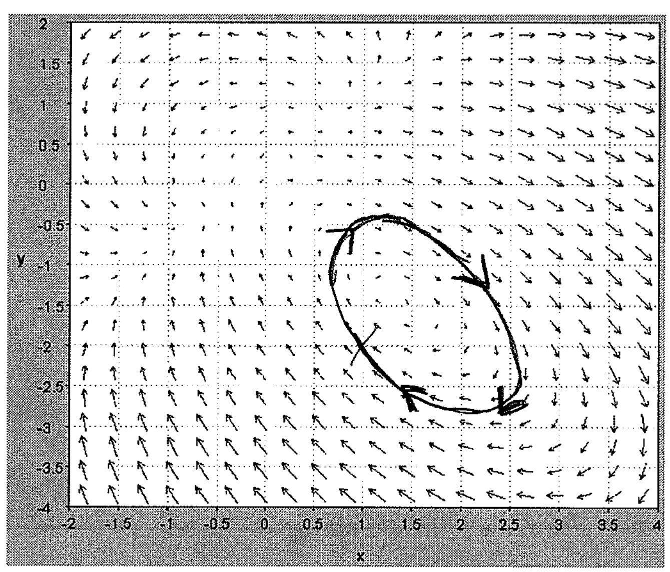
b.) Eq'n for circle is $x^2 + (y-1)^2 = 1$

let $x(t) = \cos t$

$y(t) = 1 + \sin t$

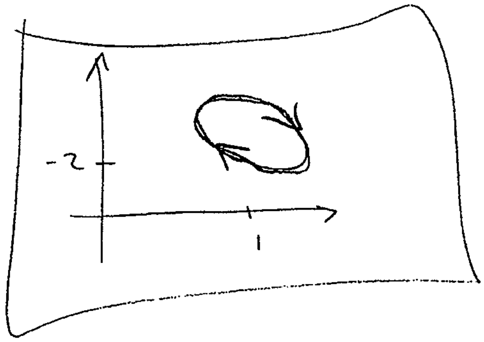
$\Rightarrow$ path is $r(t) = (\cos t)\hat{\imath} + (1+\sin t)\hat{\jmath}$ for $t \in [\pi, \frac{3\pi}{2}]$

7. (15 points)Consider the graph of *the gradient of an electric* **potential** *function.* You place a negatively charged test particle at $(1,-2)$. Sketch the particle and its approximate path. Indicate the direction of motion.



8. (15 points) Locate and identify all critical points of the function $f(x,y) = 2x - x^2 + 2y^2 - y^4$.

7.



8.) $F(x,y) = 2x - x^2 + 2y^2 - y^4$

$f_x = 2 - 2x \quad = 2(1-x) \quad \Rightarrow f_x = 0 \quad \text{if } x = 1$

$f_y = 4y - 4y^3 = 4y(1-y^2) \quad \Rightarrow f_y = 0 \quad \text{if } y = \pm 1 \text{ or } y = 0$

$f_{xx} = -2$

$f_{yy} = 4 - 12y^2$ @ $(1,-1)$

$f_{xx} f_{yy} - f_{xy}^2 \doteq f_{xx} f_{yy}$ $\quad f_{xx} < 0,\ f_{yy} < 0 \quad \Rightarrow \boxed{\cancel{\text{max @}}\ (1,-1)}$

@ $(1,0)$ $\quad f_{xx} < 0,\ f_{yy} > 0 \quad \Rightarrow \boxed{\text{saddle @ } (1,0)}$

@ $(1,1)$ $\quad f_{xx} < 0,\ f_{yy} < 0 \quad \Rightarrow \boxed{\text{max @ } (1,1)}$

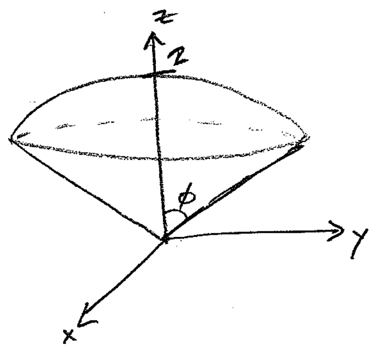
9.) $\displaystyle \lim_{(x,y)\to(1,2)} \frac{3(y-2)(x-1)^2}{(x-1)^2 + (y-2)^2}$ $\quad$ let $x = 1 + r\cos\theta$, $y = 2 + r\sin\theta$

$$= \lim_{r\to 0} \frac{3(r\sin\theta)(r\cos\theta)^2}{(r\cos\theta)^2 + (r\sin\theta)^2} = \lim_{r\to 0} \frac{3r^3 \sin\theta\cos^2\theta}{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= \lim_{r\to 0} 3r \sin\theta\cos^2\theta \quad \boxed{= 0}$$

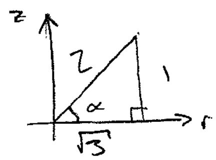
10.) a) solid bndd above by $x^2+y^2+z^2 = 4$

" " below by $z = \sqrt{\frac{x^2}{3} + \frac{y^2}{3}}$



b.) To determine ϕ, one way, write

$z = \frac{1}{\sqrt{3}} r$ and observe slope is $\frac{1}{\sqrt{3}}$ in rz-space



so hypotenuse is 2

$\Rightarrow \alpha = \pi/6$

$\Rightarrow \phi = \pi/3$

so surface $z = \sqrt{\frac{x^2}{3} + \frac{y^2}{3}}$ can be written as

$z = \frac{1}{\sqrt{3}} r$ in cylindrical coords

or $\phi = \pi/3$ in spherical coords

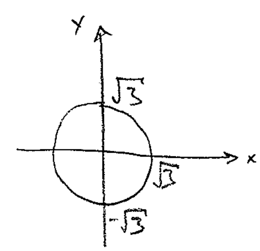
Also surface $x^2+y^2+z^2=4$ can be written as

$z = \pm\sqrt{4 - r^2}$ (will take + since $z>0$ in problem)
in cylindrical coords

or $\rho = 2$ in spherical coords

To determine extent in x and y look @ intersection

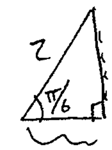
$z^2 = \frac{x^2+y^2}{3} = 4 - x^2 - y^2$



$\Rightarrow \frac{4}{3}x^2 + \frac{4}{3}y^2 = 4 \iff x^2+y^2 = 3$

so volume $= \boxed{\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{\sqrt{\frac{x^2}{3}+\frac{y^2}{3}}}^{\sqrt{4-x^2-y^2}} dz\, dy\, dx}$

c) $\boxed{\int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\frac{1}{\sqrt{3}}r}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta}$



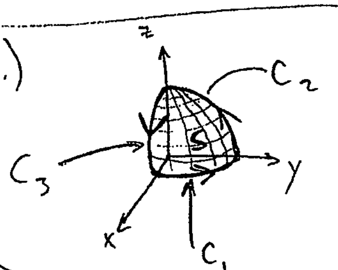
$2\cos\frac{\pi}{6} = \sqrt{3}$

d) $\boxed{\int_0^{2\pi}\int_0^{\pi/3}\int_0^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$

e) $$\int_0^{2\pi}\int_0^{\pi/3}\int_0^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/3} \sin\phi\, d\phi\right)\left(\int_0^{2} \rho^2 d\rho\right)$$

$$= 2\pi \left(-\cos\phi \Big|_0^{\pi/3}\right)\left(\frac{\rho^3}{3}\Big|_0^{2}\right] = 2\pi\left(\frac{8}{3}\right)^{-}\left(\frac{1}{2}-1\right) = 2\pi\left(\frac{8}{3}\right)\left(\frac{1}{2}\right) = \boxed{\frac{8\pi}{3}}$$

11.) a.)



b.)
C_1 : $\underline{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + 0\hat{k}$

C_2 : $\underline{r}(t) = 0 + \cos t\,\hat{\jmath} + \sin t\,\hat{k}$

C_3 : $\underline{r}(t) = \sin t + 0\hat{\jmath} + \cos t\,\hat{k}$

For $t \in 0, \pi/2$

c.) $\underline{G} = xy^2\,\hat{\imath} + yz^2\,\hat{\jmath} + x^2z\,\hat{k}$

$$\nabla\times\underline{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xy^2 & yz^2 & x^2z \end{vmatrix}$$

$$\nabla \times \underline{G} = (0 - 2yz)\hat{i} - (2xz - 0)\hat{j} + (0 - 2xy)\hat{k} = \boxed{-2(yz\hat{i} + xz\hat{j} + xy\hat{k})}$$

d.) $\underline{F} = \nabla \times \underline{G}$ The flux of $\underline{F}$ through S is $\iint_S \underline{F} \cdot \hat{n}\, d\sigma = \iint_S (\nabla \times \underline{G}) \cdot \hat{n}\, d\sigma$

Stokes says

$$\iint_S (\nabla \times \underline{G}) \cdot \hat{n}\, d\sigma = \oint_C \underline{G} \cdot d\underline{r}$$

$$\therefore \boxed{\iint_S \underline{F} \cdot \hat{n}\, d\sigma = \oint \underline{G} \cdot d\underline{r}}$$

e.) $$\oint_C \underline{G} \cdot d\underline{r} = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

on C_1: $\underline{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 0\hat{k} \Rightarrow \frac{d\underline{r}}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0k$

$$\Rightarrow \underline{G} \cdot \frac{d\underline{r}}{dt} = (xy^2\hat{i} + yz^2\hat{j} + x^2z\hat{k}) \cdot (-\sin t\,\hat{i} + \cos t\,\hat{j} + 0\hat{k})$$

$$= -(xy^2)\sin t + \underbrace{(yz^2)}_{=0}\cos t \qquad = -(\cos t)(\sin t)^2 \sin t$$

$=0$ since $z=0$ on C_1

$$= -\cos t \sin^3 t\, dt \qquad \Rightarrow \int_{C_1} \underline{G} \cdot \frac{d\underline{r}}{dt}\, dt = \int_0^{\pi/2} -\cos t \sin^3 t\, dt$$

$$= -\left.\frac{\sin^4 t}{4}\right|_0^{\pi/2} = -\frac{1}{4}$$

on C_2: $\underline{r}(t) = 0\hat{i} + \cos t\,\hat{j} + \sin t\,\hat{k} \Rightarrow \frac{d\underline{r}}{dt} = 0\hat{i} - \sin t\,\hat{j} + \cos t\,\hat{k}$

$$\Rightarrow \underline{G} \cdot \frac{d\underline{r}}{dt} = 0 - (yz^2)(\sin t) + \underbrace{\cancel{(x^2z)}}_{=0}\cos t = -(yz^2)\sin t = -\cos t \sin^3 t$$

using result

$$\Rightarrow \int_{C_2} \underline{G} \cdot \frac{d\underline{r}}{dt}\, dt = \int_0^{\pi/2} -\cos t \sin^3 t\, dt = -\frac{1}{4}$$

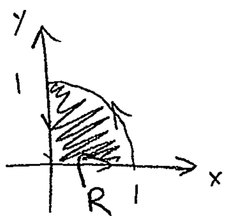
on C_3: $\underline{r}(t) = \sin t\,\hat{i} + 0\hat{j} + \cos t\,\hat{k} \Rightarrow \frac{d\underline{r}}{dt} = \cos t\,\hat{i} + 0\hat{j} - \sin t\,\hat{k}$

$$\Rightarrow \underline{G} \cdot \frac{d\underline{r}}{dt} = \underbrace{\cancel{(xy^2)}}_{0}\cos t + 0 - (x^2z)\sin t = -x^2z\sin t = -\cos t \sin^3 t$$

$$\Rightarrow \int_{C_3} \underline{G} \cdot \frac{d\underline{r}}{dt}\, dt = \int_0^{\pi/2} -\cos t \sin^3 t\, dt = -\frac{1}{4} \qquad \Rightarrow \oint_C = -\frac{1}{4} - \frac{1}{4} - \frac{1}{4} = \boxed{-\frac{3}{4}}$$

f) Flux $= \iint_S \underline{F} \cdot \hat{n}\, d\sigma = \iint_S -2(yz\hat{i} + xz\hat{j} + xy\hat{k}) \cdot \hat{n}\, d\sigma$

$$= \iint_R \pm \frac{\underline{F} \cdot \nabla g}{|\nabla g \cdot \hat{p}|}\, dA$$



let $g(x,y,z) = x^2 + y^2 + z^2 = 1$ define S, then

$\nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\Rightarrow \underline{F} \cdot \nabla g = -4(xyz + xyz + xyz) = -12xyz$

$|\nabla g \cdot \hat{p}| = |\nabla g \cdot \hat{k}| = |2z| = 2z$ since $z \geq 0$

$\Rightarrow$

$\Rightarrow$ Flux $= \iint_R \frac{-12xyz}{2z}\, dA = -6 \iint_R xy\, dA \longrightarrow$ polar

$$= -6 \int_0^{\pi/2} \int_0^1 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta = -6 \int_0^{\pi/2} \frac{r^4}{4}\Big|_0^1 \sin\theta\cos\theta\, d\theta$$

$$= -\frac{3}{2} \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = -\frac{3}{2} \frac{\sin^2\theta}{2}\Big|_0^{\pi/2} = -\frac{3}{4}$$