Exam 1 Solutions

APPM 2350, Calculus 3, Fall 2008

September 26, 2008

1. (a) False. The range of $\sin^{-1}(x)$ with positive argument, x, is $[0, \pi/2]$. The angle, θ, can be anywhere between $[0, \pi]$ for two general vectors.

   (b) True. See definition on page 808.

   (c) False. $(\mathbf{A} \cdot \mathbf{B})$ is a scalar and therefore $(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}$ only can make sense as a scalar-vector product. The same thing with $\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$. Even then, the two sides are not equal.

$$\begin{aligned} (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} &= (a_1 b_1 + a_2 b_2 + a_3 b_3)(c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}) \neq \\ \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) &= (b_1 c_1 + b_2 c_2 + b_3 c_3)(a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \end{aligned}$$

   (d) True.

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (-\mathbf{B} \times \mathbf{A}) \cdot (-\mathbf{D} \times \mathbf{C}) \\ &= (-1)^2 (\mathbf{B} \times \mathbf{A}) \cdot (\mathbf{D} \times \mathbf{C}) \\ &= (\mathbf{D} \times \mathbf{C}) \cdot (\mathbf{B} \times \mathbf{A}) \end{aligned}$$

   (e) True. $\mathbf{A} \times \mathbf{A} = \mathbf{0}$.

   (f) True. $\mathbf{A} \times \mathbf{B}$ is orthogonal to $\mathbf{A}$.

   (g) True. Both sides are zero because $\mathbf{A} \times \mathbf{B}$ is orthogonal to both $\mathbf{A}$ and $\mathbf{B}$.

   (h) True. By the scalar-vector product rule of derivatives:

$$\begin{aligned} \frac{d}{dt}\left[c^2(t)\mathbf{r}(t)\right] &= \frac{d}{dt}[c^2(t)]\mathbf{r}(t) + c^2(t)\frac{d}{dt}\mathbf{r}(t) \\ &= 2c'(t)c(t)\mathbf{r}(t) + c^2(t)\frac{d}{dt}\mathbf{r}(t) \\ &= c(t)\left[2c'(t)\mathbf{r}(t) + c(t)\frac{d}{dt}\mathbf{r}(t)\right] \end{aligned}$$

(i) False. The cross product does not commute. By the cross-product rule of derivatives:

$$\frac{d}{dt}\left[\mathbf{r}(t) \times \mathbf{s}(t)\right] \quad = \quad \frac{d}{dt}\mathbf{r}(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \frac{d}{dt}\mathbf{s}(t)$$

(j) False. Suppose $\mathbf{A} = \mathbf{i}$ and $\mathbf{B} = \mathbf{j}$. Then,

$$\begin{aligned} \mathbf{i} \times (\mathbf{i} \times (\mathbf{i} \times (\mathbf{i} \times (\mathbf{i} \times \mathbf{j})))) &= \\ \mathbf{i} \times (\mathbf{i} \times (\mathbf{i} \times (\mathbf{i} \times \mathbf{k}))) &= \\ \mathbf{i} \times (\mathbf{i} \times (\mathbf{i} \times (-\mathbf{j}))) &= \\ \mathbf{i} \times (\mathbf{i} \times (-\mathbf{k})) &= \\ \mathbf{i} \times \mathbf{j} &= \mathbf{k}. \end{aligned} \tag{1}$$

2. $\mathbf{r}(t)$ is such that $|\mathbf{v}(t)| = 1$.

(a)

$$L = \int_{t_0}^{t_1} |\mathbf{v}(t)|\, dt = [t]_{t_0}^{t_1} = t_1 - t_0. \tag{2}$$

(b) Note that $|\mathbf{v}(t)|^2 = 1$. Then,

$$0 = \frac{d}{dt}\left(|\mathbf{v}(t)|^2\right) = \frac{d}{dt}\left(\mathbf{v}(t) \cdot \mathbf{v}(t)\right) = \mathbf{a}(t) \cdot \mathbf{v}(t) + \mathbf{v}(t) \cdot \mathbf{a}(t) = 2\mathbf{a}(t) \cdot \mathbf{v}(t).$$

This shows $\mathbf{a}(t) \perp \mathbf{v}(t)$.

(c) $a_T(t) = \frac{d}{dt}|\mathbf{v}(t)| = 0$ and $a_N(t) = \sqrt{|\mathbf{a}(t)|^2 - a_T^2(t)} = |\mathbf{a}(t)|$.

3. $A(0,0,1)$, $B(3,0,0)$, and $C(0,2,0)$.

(a) Vectors $\vec{AB} = 3\mathbf{i} - \mathbf{k}$ and $\vec{AC} = 2\mathbf{j} - \mathbf{k}$, so

$$\vec{AB} \times \vec{AC} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}. \tag{3}$$

Let this vector be $\mathbf{n}$, the normal component to plane S. Then, S is written $2x + 3y + 6z = d$. Point A (or B, or C) shows $d = 6$.

(b) $D(0,0,0)$, so $\vec{AD} = -\mathbf{k}$.

$$\vec{ED} = \mathrm{proj}_{\mathbf{n}}\vec{AD} = \frac{(\vec{AD} \cdot \mathbf{n})}{|\mathbf{n}|^2}\mathbf{n} = -\frac{6}{49}(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) \tag{4}$$

Therefore, point E is found by moving from D up $\vec{ED}$ backwards:

$$E\left(\frac{12}{49}, \frac{18}{49}, \frac{36}{49}\right). \tag{5}$$

An Alternate Solution: Observe a line through D in the direction of $\mathbf{n}$ goes through plane S at the point of minimum distance, E. This line is parametrized by $x(t) = 2t$, $y(t) = 3t$, and $z(t) = 6t$. To find where this line crosses S, put the line coordinates into the plane equation and solve for t:

$$2(2t) + 3(3t) + 6(6t) = 6 \qquad \Longrightarrow \qquad t = \frac{6}{49}. \tag{6}$$

Then, plugging t into the parameterized equations gives the same answer for E.

(c) $\vec{AF} = \text{proj}_{\vec{AB}}\vec{AD} = \frac{(\vec{AD}\cdot\vec{AB})}{|\vec{AB}|^2}\vec{AB} = \frac{1}{10}(3\mathbf{i} - \mathbf{k})$. Moving to F from A gives

$$F\left(\frac{3}{10}, 0, \frac{9}{10}\right). \tag{7}$$

An Alternate Solution: Observe that D, B, and A are all in the x-z plane. The line that goes through D that has direction orthogonal to $\vec{AB}$ (and in the x-z plane) goes through plane S at the point of minimal distance, F. This line is parametrized by $x(t) = t$, $y(t) = 0$, and $z(t) = 3t$. Then, plug the coordinate equations of this line into the equation for S to find the point of intersection:

$$2(t) + 3(0) + 6(3t) = 6 \qquad \Longrightarrow \qquad t = \frac{3}{10}. \tag{8}$$

Plugging t back into the coordinate equations give the same answer for F.

(d) We know the line must go through point D, so we must find a direction for the line. The $\mathbf{i}$- and $\mathbf{j}$-components are equal, so the direction is of the form $\mathbf{w} = p\mathbf{i} + p\mathbf{j} + q\mathbf{k}$ for scalars p and q. Also the line should be moving in a plane parallel to S, or $\mathbf{w} \perp \mathbf{n}$. Simplifying,

$$\begin{aligned} (p\mathbf{i} + p\mathbf{j} + q\mathbf{k}) \cdot (2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) &= \\ 5p + 6q &= 0. \end{aligned}$$

The length of $\mathbf{w}$ does not matter (as long as it is not the zero vector), so we choose $q = -5$. This means that $p = 6$ (by the derived equation), and $\mathbf{w} = 6\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$. Then the line is parameterized by

$$L = \begin{cases} x(t) &= 6t \\ y(t) &= 6t \\ z(t) &= -5t \end{cases}, \qquad -\infty < t < \infty. \tag{9}$$

4. We start by computing the acceleration and velocity:

$$\begin{array}{rcrcr} \mathbf{r}(t) & = & (2t+3)\,\mathbf{i} & + & (5-t^2)\,\mathbf{j} \\ \mathbf{v}(t) & = & 2\,\mathbf{i} & & -2t\,\mathbf{j} \\ \mathbf{a}(t) & = & & & -2\,\mathbf{j} \end{array} \tag{10}$$

$$\hat{\mathbf{T}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+t^2}}(\mathbf{i} - t\,\mathbf{j}) \tag{11}$$

We know that $\hat{\mathbf{B}} = -\mathbf{k}$ and torsion $\tau = 0$ because this is a clock-wise moving curve in the x-y plane.

$$\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}} = \frac{1}{\sqrt{1+t^2}}(-t\,\mathbf{i} - \mathbf{j}). \tag{12}$$

$$\kappa = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{4}{(4+4t^2)^{3/2}} = \frac{1}{2(1+t^2)^{3/2}}. \tag{13}$$

5. (a) This is a one-variable limit, so we may use L'Hopital's rule ($\Rightarrow$):

$$\frac{1-\cos x}{\tan^2 x} \underset{L'H.}{\Longrightarrow} \frac{\sin x}{2\tan x \sec^2 x} = \frac{\cos^3 x}{2} \longrightarrow \frac{1}{2} \quad \text{as} \quad x \to 0. \tag{14}$$

(b) Move to $(0,0)$ along line $y = kx$ for some k:

$$\frac{(x+y)^2}{x^2+y^2} = \frac{(x+kx)^2}{x^2+k^2x^2} = \frac{(1+k)^2x^2}{(1+k^2)x^2} = \frac{1+2k+k^2}{1+k^2}. \tag{15}$$

If $k = 0$, then the functions goes to 1 as $(x,y) \to (0,0)$. For $k = 1$, the function goes to 2. The function approaches different values for different paths through $(0,0)$ so the limit does not exist by the *the two-path test* (see pg. 920).

(c) The top can be factored and one term cancels with the bottom:

$$\frac{xy+x+2y+2}{y+1} = \frac{(y+1)(x+2)}{y+1} \longrightarrow 3 \qquad \text{as} \qquad (x,y) \to (1,-1). \tag{16}$$