Exam 2 Solutions

APPM 2350, Calculus 3, Fall 2008

October 20, 2008

1. Answers in table form:

a.	b.	c.	d.	e.	f.	g.	h.	i.	j.
B	C	C	D	B	A	D	B	D	B

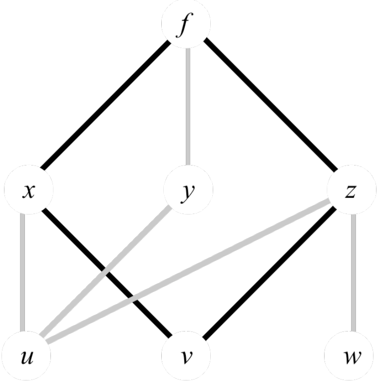


Figure 1
(Problem 1a)

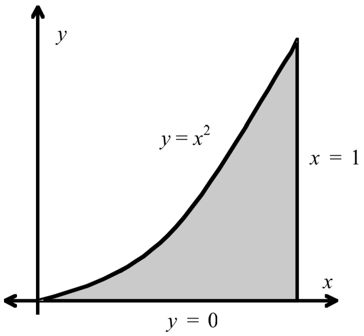


Figure 2
(Problem 1e)

(a) The full dependency graph is given in Figure 1. Using all paths from f to v and summing the terms from each path gives B as the answer.

(b) The equation $f_y = 0$ must hold at a critical point by the First Derivative Test. For this function $f_y = e^x = 0$, which does not hold for any $-\infty < x < \infty$. Thus, the answer is C.

(c) Let $g(x, y, z) = f(x, y) - z$, then ∇g is normal to the $g = 0$ level surface at any point on the surface. In terms of f, $\nabla g = f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k}$, so the answer is C.

(d) Let $y = -x$ for values of $x \neq 0$. This function does not exist along this path, thus, according to the definition of a multivariable limit, the limit as $(x, y) \to (0, 0)$ does not exist, either. A Two Path Test is more difficult: first, consider $y = x$. The limit of $2x^2/x$ as $x \to 0$ is 0. Then we must create a path that exposes the problem in the denominator. The curve $y = -x + x^2$ is such a path.

$$\lim_{x\to 0} \frac{x^2 + (-x + x^2)^2}{x + (-x + x^2)} = \lim_{x\to 0} \frac{2x^2 - 2x^3 + x^4}{x^2} = \lim_{x\to 0}(2 - 2x + x^2) = 2. \qquad (1)$$

Theferore, D is the answer.

(e) The integral is computed on the region in Figure 2. To integrate in y first, consider a fixed x, and then note that a vertical line enters the region at $y = 0$ and exits at $y = x^2$. The smallest x-value in the region is 0 and the largest is 1. To switch the order of integration properly, we must choose B.

(f) See the box on page 960; the answer is A.

(g) The linearization formula is: $L(x, y) = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)$. We calculate,

$$\begin{aligned} f(1,1) &= (1)^2 + (1)^2 + 1 = 3, \\ f_x(1,1) &= 2x\Big|_{x=1} = 2, \\ f_y(1,1) &= 2y\Big|_{y=1} = 2. \end{aligned}$$

Then, $L(x, y) = 3 + 2(x - 1) + 2(y - 1)$, which simplifies to D.

(h) Set $f(x, y) = xy$, then $f_x = y$ and $f_y = x$. The differential formula gives

$$df = f_x\, dx + f_y\, dy = y\, dx + x\, dy. \tag{2}$$

Dividing by $f = xy$ predicts the relative uncertainty:

$$\frac{df}{f} = \frac{y\, dx}{xy} + \frac{x\, dy}{xy} = \frac{dx}{x} + \frac{dy}{y}. \tag{3}$$

This shows that 1% uncertainty in x and y gives approximately 2% uncertainty in f; the answer is B.

(i) Set $f(x, y) = x + y$, then $f_x = 1$ and $f_y = 1$. The differential formula gives

$$df = f_x\, dx + f_y\, dy = dx + dy. \tag{4}$$

Dividing by $f = x + y$ predicts the relative uncertainty:

$$\frac{df}{f} = \frac{dx}{x+y} + \frac{dy}{x+y} = \frac{dx}{x}\left(\frac{x}{x+y}\right) + \frac{dy}{y}\left(\frac{y}{x+y}\right). \tag{5}$$

This shows that $x + y$ very small compared to either x or y yields arbitrarily large uncertainty. For actual examples consider $x = 101$ and $y = -100$ with $dx = 1, dy = -1$ (approximately 1% for each). Then the relative uncertainty is predicted to be $df/f = .01(101) + (-.01)(-100) = 2.01$ or 201%. The same calculation for $x = 100.1$, $y = -100$, and $dx = 1, dy = -1$ gives $df/f = .01(1001) + (-.01)(-1000) = 20.01$ or 2001%. This can be generalized to arbitrary uncertainty in f, so the answer is D.

(j) Plugging $x = y^2$ into $f(x, y)$ gives a single-variable function

$$g(y) = f(y^2, y) = 3y - y^2 \tag{6}$$

Use the First Derivative Test of single-variable calculus exposes a single critical point:

$$g'(y) = 3 - 2y = 0 \qquad \Longrightarrow \qquad y = \frac{3}{2}. \tag{7}$$

The Second Derivative Test of single-variable calculus shows this point is a maximum, $g''(y) = -2 < 0$. As $y \to \pm\infty$, $g \to -\infty$. Pick B.

2. Find and classify critical points of $f(x, y) = \frac{1}{3}x^3 + xy^2 - x$. Use the First Derivative Test to set up the following nonlinear system of equations:

$$\begin{array}{rcl} f_x & = & 0 \\ f_y & = & 0 \end{array} \quad \begin{array}{c} \Longrightarrow \\ \Longrightarrow \end{array} \quad \begin{array}{lrcl} \text{(i)} & x^2 + y^2 - 1 & = & 0 \\ \text{(ii)} & 2xy & = & 0. \end{array} \tag{8}$$

From equation (ii), we have two cases: either $x = 0$ or $y = 0$. If $x = 0$, then equation (i) simplifies to $y^2 = 1$, meaning either $y = 1$ or $y = -1$. If $y = 0$, then equation (i) simplifies to $x^2 = 1$, meaning either $x = 1$ or $x = -1$. In summary, there are four critical points:

$$\Big\{ (1, 0),\ (-1, 0),\ (0, 1),\ (0, -1) \Big\}. \tag{9}$$

To classify them, use the Second Derivative Test. The second-order partials are

$$f_{xx} = 2x \qquad f_{yy} = 2x \qquad f_{xy} = 2y, \tag{10}$$

and the discriminant is

$$f_{xx}f_{yy} - f_{xy}^2 = 4x^2 - 4y^2. \tag{11}$$

The following table gives the results of the Second Derivative Test:

Point	Discriminant	f_{xx}	Classification
$(1, 0)$	4	2	Local Min
$(-1, 0)$	4	-2	Local Max
$(0, 1)$	-4		Saddle
$(0, -1)$	-4		Saddle

(12)

3. Minimize $f(x, y, z) = x^2 + (y + z)^2$ subject to constraint $g(x, y, z) = x + 2y = 1$.

(a) We set up the Lagrange Multiplier system, $\nabla f = \lambda \nabla g$, including the constraint:

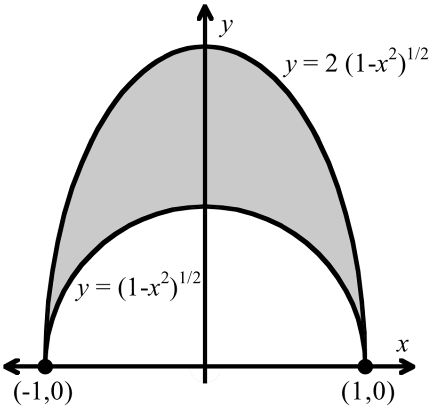
$$\begin{array}{rcl} f_x & = & \lambda g_x \\ f_y & = & \lambda g_y \\ f_z & = & \lambda g_z \\ g & = & 1 \end{array} \quad \begin{array}{c} \Longrightarrow \\ \Longrightarrow \\ \Longrightarrow \\ \Longrightarrow \end{array} \quad \begin{array}{lrcl} \text{(i)} & 2x & = & \lambda \\ \text{(ii)} & 2(y + z) & = & 2\lambda \\ \text{(iii)} & 2(y + z) & = & 0 \\ \text{(iv)} & x + 2y & = & 1 \end{array} \tag{13}$$

(b) From equation (iii) we see that $y = -z$. Using this in equation (ii) shows $\lambda = 0$, which plugged into equation (i) gives $x = 0$. Lastly, equation (iv) gives $y = 1/2$, so $z = -1/2$. The unique solution to equations (i)-(iv) is

$$\left\{ (x, y, z) = \left(0, \frac{1}{2}, -\frac{1}{2}\right), \lambda = 0 \right\}. \tag{14}$$

(c) $f(0, 1/2, -1/2) = 0^2 + (1/2 - 1/2)^2 = 0$ is the minimum.

4. (a) The sketch is given below:



(b) Compute the volume with an iterated integral:

$$\begin{aligned}
\int\!\!\int_R f(x,y)\,dA &= \int_{x=-1}^{x=1} \left(\int_{y=\sqrt{1-x^2}}^{y=2\sqrt{1-x^2}} y\,dy \right) dx = \int_{x=-1}^{x=1} \left(\frac{y^2}{2} \right)_{y=\sqrt{1-x^2}}^{y=2\sqrt{1-x^2}} dx \\
&= \frac{1}{2} \int_{x=-1}^{x=1} \left(4(1-x^2) - (1-x^2) \right) dx \\
&= \frac{3}{2} \int_{x=-1}^{x=1} 1 - x^2\,dx = \frac{3}{2} \left(x - \frac{x^3}{3} \right)_{x=-1}^{x=1} \\
&= \frac{3}{2} \left(\left(1 - \frac{1}{3} \right) - \left(-1 + \frac{1}{3} \right) \right) \quad = \quad 2.
\end{aligned}$$