Final Solutions

APPM 2350, Calculus 3, Fall 2008

December 17, 2008

1. Answers in table form:

a.	b.	c.	d.	e.	f.	g.	h.	i.	j.
C	B	C	D	A	C	D	D	A	B

(a) Implicit differentation gives $a + b\frac{dy}{dx} = 0$. Solving for the slope gives $\frac{dy}{dx} = -\frac{a}{b}$. The normal slope is $\frac{b}{a}$, so answer C.

(b) This is B.

(c) The ellipse is parametrized with $x(t) = a\cos t$ and $y(t) = b\sin t$ from $0 \le t \le 2\pi$. The arc length of this curve is given in answer C.

(d) Consider the set of paths, $y = kx^2 - x$, for constants $k \neq 0$. Then,

$$\frac{xy}{x+y} = \frac{kx^3 - x^2}{kx^2} = x - \frac{1}{k}, \tag{1}$$

which goes to -1/k as $x \to 0$. By the *two-path test*, this limit does not exist. Pick D.

(e) As a function of t, $f(x(t), y(t)) = \sin^2(t^2) + \cos^2(t^2) = 1$, and its differential is $df = 0dt$. Therefore there is no uncertainty. Sorry Heisenberg, its A.

(f) For $f(x,y,z) = \sqrt{xyz}$, we have $f_x = \frac{\sqrt{yz}}{2\sqrt{x}}$, $f_y = \frac{\sqrt{xz}}{2\sqrt{y}}$, and $f_z = \frac{\sqrt{xy}}{2\sqrt{z}}$. The linearization about (1,1,1) is given by

$$\begin{aligned} f(1,1,1) + f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1) &= \\ 1 + \frac{1}{2}(x+y+z-3) = -\frac{1}{2}(1-x-y-z) \end{aligned}$$

This matches C.

(g) $f(x,y) = (\tan x)(\cos y)$, $\nabla f = \sec^2 x \cos y\, \mathbf{i} - \tan x \sin y\, \mathbf{j}$ at $(0,0)$ the direction of fastest decrease is $-\nabla f(0,0) = -\mathbf{i}$, which has $\theta = \pi$, pick D.

(h) Switching the ordering gives D.

(i) This is A.

(j) Answer B passes the test $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, and its potential function is $f(x,y) = \frac{x^2y^2}{2}$.

2. The curve is given by $\mathcal{C} : \{x(t) = t^2, \quad y(t) = t^3 - t, \quad -1 \le t \le 1\}$.

   (a) We have $x'(t) = 2t$ and $y'(t) = 3t^2 - 1$, so $x'(1/2) = 1$ and $y'(1/2) = -1/4$. Therefore, $\frac{dy}{dx} = y'(1/2)/x'(1/2) = -1/4$. The location of the point is $(x(1/2), y(1/2)) = (1/4, -3/8)$. The equations are:

   $$\begin{array}{rrcl} \text{Tangent:} & (y + 3/8) & = & -\frac{1}{4}(x - 1/4) \\ \text{Normal:} & (y + 3/8) & = & 4(x - 1/4) \end{array}$$

   (b) Set $g(t) = f(x(t), y(t)) = t^2 + t^3 - t$. Then, use the single-variable first-derivative test: $g'(t) = 2t + 3t^2 - 1 = 0$. The solutions to this quadratic equation are $t = \{-1, 1/3\}$, but $t = -1$ is not on the interior of our interval. A second derivative test, $g''(t) = 6t + 2$, $g''(1/3) = 4$, shows there is a local minimum at $t = 1/3$. We also need to test the values of the endpoints $t = -1$ and $t = 1$.

t	$x(t)$	$y(t)$	$f(x(t), y(t))$	Classification of Point
-1	1	0	1	Global Maximum
1/3	1/9	-8/27	-5/27	Global Minimum
1	1	0	1	Global Maximum

   (c) Using $\boxed{\text{Green's Theorem}}$ on the field $\mathbf{F} = yj$, we have the following area formula:

$$\int\int_R dA = \int\int_R \frac{\partial N}{\partial y}\, dA = -\int_C N\, dx = -\int_C y\, dx. \tag{2}$$

$$-\int_C y\, dx = -\int_{-1}^{1} (t^3 - t)(2t)\, dt = -2\left(\frac{t^5}{5} - \frac{t^3}{3}\right)_{-1}^{1} = \frac{8}{15}. \tag{3}$$

3. (a) We apply $\boxed{\text{Stoke's Theorem}}$ and calculate the circulation of $\mathbf{F}$ around C instead.

$$\begin{array}{rcrcrcr} \mathbf{r}(t) & = & \cos t\,\mathbf{i} & + & \sin t\,\mathbf{j} & + & \mathbf{k} \\ \mathbf{v}(t) & = & -\sin t\,\mathbf{i} & + & \cos t\,\mathbf{j} & + & 0\,\mathbf{k} \end{array} \tag{4}$$

$$\begin{aligned} \int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_C M\,dx + N\,dy + P\,dz \\ &= \int_0^{2\pi} (-\sin t)(-\sin t) + (\cos t)(\cos t) + (1)(0)\,dt \\ &= \int_0^{2\pi} 1\,dt = 2\pi. \end{aligned}$$

(b) Instead of calculating the surface-flux directly, we apply the $\boxed{\text{Divergence Theorem}}$. First, we must make a closed surface by introducing $\mathcal{S}_2$, the plane $z = 1$. Then, from the the divergence theorem, we have:

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int\int\int_D \nabla\cdot\mathbf{F}\,dV - \int\int_{S_2} \mathbf{F}\cdot\mathbf{n}_2\,d\sigma_2. \tag{5}$$

Notice that $\nabla\cdot\mathbf{F} = 0+0+0 = 0$, so the triple integral above is also 0. We only have to evaluate the surface-flux integral on $\mathcal{S}_2$. The surface $f(x,y,z) = z = 1$ is most easily integrated in the xy plane, so $\mathbf{p} = \mathbf{k}$. Calculate $\nabla f = \mathbf{k}$, and set the normal outward, $\mathbf{n}_2 = -\nabla f = -\mathbf{k}$. Then, apply the surface-flux integration formula:

$$\begin{aligned} \int\int_{S_2} \mathbf{F}\cdot\mathbf{n}_2\,d\sigma_2 &= \int\int_R \frac{\mathbf{F}\cdot(-\nabla f)}{|\nabla\cdot\mathbf{k}|}\,dA \\ &= \int\int_R \frac{-1}{1}\,dA = -\int\int_R dA = -\pi, \end{aligned}$$

because R is the unit circle. Then,

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = (0) - (-\pi) = \pi. \tag{6}$$

4. (a)

$$\frac{dP}{dt} = \frac{\partial P}{\partial g}\frac{dz}{dt} + \frac{\partial P}{\partial T}\left(\frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}\right) + \frac{\partial P}{\partial z}\frac{dz}{dt}, \tag{7}$$

$$\left.\frac{dP}{dt}\right|_{t_1} = (6)(5) + (0)\Big((1)(3) + (0)(4) + (2)(0)\Big) + (7)(0) = 30. \tag{8}$$

(b)

$$\frac{dP}{ds} = \frac{dP}{dt}\frac{dt}{ds} = \frac{1}{|\mathbf{v}|}\frac{dP}{dt}, \tag{9}$$

$$\left.\frac{dP}{ds}\right|_{t_1} = \frac{30}{\sqrt{3^2+4^2}} = 6. \tag{10}$$

(c) $\mathbf{u} = -\nabla T = -\mathbf{j} - 2\mathbf{k}$ and set the other vector orthogonal to ∇T, $\mathbf{w} = \mathbf{i}$ or $\mathbf{w} = 2\mathbf{j} - \mathbf{k}$.

5. $f(x,y) = 2xy + 1$

   (a) The two-variable, first derivative test gives the linear system

$$\begin{array}{lll} (1) & f_x = 0 & : \quad 2y = 0 \\ (2) & f_y = 0 & : \quad 2x = 0 \end{array} \tag{11}$$

   Equation (1) implies $y = 0$ and Equation (2) shows $x = 0$. Therefore the only critical point is $(0,0)$. To classify this point, use the second derivative test. We have $f_{xx} = f_{yy} = 0$, $f_{xy} = 2$, and the discriminant $f_{xx}f_{yy} - f_{xy}^2 = -4$. So (0,0) is a *Saddle Point.*

   (b)

$$\begin{array}{lllll} (1) & f_x & = & 0 & : \quad 2y = \lambda 2x \\ (2) & f_y & = & 0 & : \quad 2x = \lambda 2y \end{array} \tag{12}$$

   First, solving Equation (1) for y gives $y = \lambda x$. Plugging this into Equation (2) and simplifying gives $x = \lambda^2 x$. This implies $x = 0$ or $\lambda^2 = 1$. If $x = 0$, then $y = 0$ (by either (1) or (2)), and the point $(0,0)$ is not on the constraint. Thus, $\lambda^2 = 1$, meaning either $\lambda = 1$ or $\lambda = -1$. If $\lambda = 1$, then Equation (1) implies $x = y$. Plugging this into the constraint yields $2x^2 = 1$, or $x = \pm 1/\sqrt{2}$, or $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. If $\lambda = -1$, then $x = -y$, which, plugged into the constraint, also yields $x = \pm 1/\sqrt{2}$, or $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$.

   (c) The following table shows the values of $f(x,y)$ and the classification for these five points:

(x,y)	$f(x,y)$	Classification of Point
$(0,0)$	1	Saddle Point
$(1/\sqrt{2}, 1/\sqrt{2})$	2	Global Maximum
$(1/\sqrt{2}, -1/\sqrt{2})$	0	Global Minimum
$(-1/\sqrt{2}, 1/\sqrt{2})$	0	Global Minimum
$(-1/\sqrt{2}, -1/\sqrt{2})$	2	Global Maximum

   (d) Let $g(x,y) = 2xy - z = -1$. Then $\nabla g = 2y\,\mathbf{i} + 2x\,\mathbf{j} - \mathbf{k}$ and $|\nabla g| = \sqrt{4x^2 + 4y^2 + 1}$. This surface is most easily integrated by projecting into the xy plane, so let $\mathbf{p} = \mathbf{k}$ and let R be the shadow, the unit circle, $x^2 + y^2 \leq 1$. Employing the surface integral formula gives:

$$\begin{aligned} \int\!\!\int_S d\sigma &= \int\!\!\int_R \frac{|\nabla g|}{|\nabla g \cdot \mathbf{k}|}\, dA \\ &= \int\!\!\int_R \frac{\sqrt{4x^2 + 4y^2 + 1}}{1}\, dA \end{aligned}$$

We finish up in polar coordinates:

$$\begin{aligned}\int\!\!\int_S d\sigma &= \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\, r\, dr\, d\theta \qquad (\text{Let } u = 4r^2+1)\\ &= \frac{1}{8}\int_0^{2\pi}\int_1^5 u^{1/2}\, du\, d\theta = \frac{\pi}{6}(5^{3/2}-1).\end{aligned}$$