Alternative Final Solutions

APPM 2350, Calculus 3, Fall 2008

December 17, 2008

1. Answers in table form:

a.	b.	c.	d.	e.	f.	g.	h.	i.	j.
D	C	A	B	B	A	D	D	C	C

   (a) This equation is rewritten as $(x+1)^2+(y+1)^2+(z+1)^2=3$, a sphere of radius $\sqrt{3}$. The answer is D.

   (b) A general vector normal to this plane is given by $t\,\mathbf{i}+t\,\mathbf{j}+t\,\mathbf{k}$. Moving from the origin to the plane, we plug the components in, $x=y=z=t$, to $x+y+z=1$ and get $t=1/3$. The point $(1/3,1/3,1/3)$ is $1/\sqrt{3}$ from the origin, so the answer is C.

   (c) All paths (except $y=\sqrt{x}$, which is not in the domain) make $f\to 0$, A.

   (d) The volume of a sphere is given by $V=(4/3)\pi r^3$, its derivative $dV=4\pi r^2\,dr$. Dividing through by V gives:

$$\frac{dV}{V}=\frac{4\pi r^2 dr}{(4/3)\pi r^3}=3\frac{dr}{r}.$$

   Therefore the answer is $3(1\%)=3\%$, or B.

   (e) Implicit differentation of $f(x,y(x))=0$ gives $f_x+f_y\frac{dy}{dx}=0$. Solving yields, $\frac{dy}{dx}=-f_x/f_y$. Answer B.

   (f) The only one that works is A.

   (g) Swapping the order of integration gives D.

   (h) This is D.

   (i) This is C.

   (j) The field is conservative. The work integral is $f(B)-f(A)=3$, C.

2. The curve is given by $\mathcal{C}:\{x(t)=t-t^2,\quad y(t)=t^2-t^3,\quad 0\le t\le 1\}$

   (a) We have $x'(t)=1-2t$ and $y'(t)=2t-3t^2$, so $x'(1/3)=1/3$ and $y'(1/3)=1/3$. Therefore, $\frac{dy}{dx}=y'(1/3)/x'(1/3)=1$. The location of the point is $(x(1/3),y(1/3))=(2/9,2/27)$. The equations are:

$$\begin{array}{rrcl}\text{Tangent:} & (y-2/27) & = & (x-2/9)\\ \text{Normal:} & (y-2/27) & = & -(x-2/9)\end{array}$$

(b) Set $g(t) = f(x(t), y(t)) = t - t^3$. Then, use the single-variable first-derivative test: $g'(t) = 1 - 3t^2 = 0$. The solutions to this quadratic equation are $t = \{-1/\sqrt{3}, 1/\sqrt{3}\}$, but $t = -1/\sqrt{3}$ is not on the interior of our interval. A second derivative test, $g''(t) = -6t$, $g'(1/\sqrt{3}) = -6/\sqrt{3}$, shows there is a local maximum at $t = 1/\sqrt{3}$. We also need to test the values of the endpoints $t = 0$ and $t = 1$.

t	$x(t)$	$y(t)$	$f(x(t), y(t))$	Classification of Point
0	0	0	0	Global Minimum
$1/\sqrt{3}$	$3^{-1/2} - 1/3$	$1/3 - 3^{-3/2}$	$3^{-1/2} - 3^{-3/2}$	Global Maximum
1	0	0	0	Global Minimum

(c) Using Green's Theorem on the field $\mathbf{F} = x\mathbf{i}$, we have the following area formula:

$$\int\int_R dA = \int\int_R \frac{\partial M}{\partial x}\, dA = \int_C M\, dy = \int_C x\, dy.$$

$$\int_C x\, dy = \int_0^1 (t - t^2)(2t - 3t^2)\, dt = \left(2\frac{t^3}{3} - 5\frac{t^4}{4} + 3\frac{t^5}{5}\right)_0^1 = \frac{1}{60}.$$

3. (a) First, the the curl of field $\mathbf{F} = -xy\,\mathbf{i} + xy\,\mathbf{j} + (zx - zy)\,\mathbf{k}$ is

$$\nabla \times \mathbf{F} = -z\,\mathbf{i} + -z\,\mathbf{j} + (x + y)\,\mathbf{k}.$$

By Stokes Theorem, we can evaluate $\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$ on any surface that has the unit circle in the xy plane as a boundary. There are 2 easy options: (i) use the upper unit hemisphere or (ii) use the unit circle in the xy-plane. Alternatively, we can use Stokes Theorem to (iii) evaluate the circulation around the unit circle in the xy-plane. All threee options are below.

(i) For the upper-half of the unit hemisphere, we have $f(x, y, z) = x^2 + y^2 + z^2 = 1$. The upward unit normal is given by $\mathbf{n} = \nabla f / |\nabla f| = (2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k})/|\nabla f|$. The integrand is nice, $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = 0$. Therefore the surface-curl integral is 0.

(ii) For the xy-plane, we have $f(x, y, z) = z = 0$. The upward unit normal is given as $\mathbf{n} = \nabla f / |\nabla f| = \mathbf{k}$. Then,

$$\begin{aligned}\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x + y)\, dy\, dx \\ &= \int_{-1}^{1} \left(2x\sqrt{1 - x^2}\right) dx \\ &= \left(-\frac{4}{3}(1 - x^2)^{3/2}\right)_{-1}^{1} = 0.\end{aligned}$$

(iii) The unit circle is parameterized by $\mathcal{C} : \mathbf{r} = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j} + 0\,\mathbf{k}$.

$$\begin{aligned}\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma &= \int_{\mathcal{C}} (-xy)\, dx + (xy)\, dy + (zx - zy)\, dz \\ &= \int_0^{2\pi} \left(\cos(t)\sin^2(t) + \cos^2(t)\sin(t)\right)\, dt \\ &= \left(\frac{\sin^3(t)}{3} - \frac{\cos^3(t)}{3}\right)\Bigg|_0^{2\pi} = 0\end{aligned}$$

(b) The direct computation of the outward flux requires the evaluation of six surface integrals! Even worse, finding antiderivatives of $e^{(y^2)}$ and $\cos(x^2)$ by hand is impossible. Instead, use the Divergence Theorem and calculate the triple integral of the divergence over the interior of the closed surface, $\mathcal{S}$:

$$\begin{aligned}\int\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \int\int\int_D \nabla \cdot \mathbf{F}\, dV \\ &= \int_{-1}^{1}\int_{-1}^{1}\int_0^{\sqrt{4-x^2-y^2}} 2z\, dz\, dy\, dx \\ &= \int_{-1}^{1}\int_{-1}^{1} (4 - x^2 - y^2)\, dy\, dx \\ &= \int_{-1}^{1} (8 - 2x^2 - 2/3)\, dx \\ &= 16 - 4/3 - 4/3 = 40/3\end{aligned}$$

4. This problem uses the multivariable function, $f(x, y) = y^2 - yx^2 + y$.

(a) Use the two-variable first-derivative test to get the system of nonlinear equations:

$$\begin{array}{llrcl}(1) & f_x : & -2xy & = & 0 \\ (2) & f_y : & 2y - x^2 + 1 & = & 0\end{array}$$

From Equation (1), we have either $x = 0$ or $y = 0$. If $x = 0$, then Equation (2) becomes $2y + 1 = 0$ and there is a critical point at $(0, -1/2)$. If $x = 0$, then Equation (2) becomes $-x^2 + 1 = 0$ and there are critical points at $(1, 0)$ and $(-1, 0)$.

To classify these points, use the second derivative test. We have $f_{xx} = -2y$, $f_{xy} = -2x$, and $f_{yy} = 2$. Then the discriminant is $f_{xx}f_{yy} - f_{xy}^2 = -4y - 4x^2$.

Point	f_{xx}	Discriminant	Classification
(0, -1/2)	1	8	Local Minimum
(1, 0)	0	-4	Saddle Point
(-1, 0)	0	-4	Saddle Point

(b) $\nabla f = -2xy\,\mathbf{i} + (2y - x^2 + 1)\,\mathbf{j}$ and $\mathbf{u} = -\nabla f(2,1) = 4\,\mathbf{i} + \mathbf{j}$

(c) By the chain rule, $\frac{df}{dt} = \nabla f \cdot \mathbf{v} = -14$.

5. (a) The easiest way to do this is by projecting the surface into the yz-plane. The shadow, R, is the unit circle $y^2+z^2 \leq 1$ and $\mathbf{p} = \mathbf{i}$. Let $g(z, y, z) = -x+y^2+z^2 = 0$, then $\nabla g = -\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$ and $|\nabla g| = \sqrt{1 + 4y^2 + 4z^2}$. The surface-integral formula gives:

$$\int\int_S d\sigma = \int\int_R \frac{|\nabla g|}{|\nabla g \cdot \mathbf{i}|}\,dA = \int\int_R \frac{\sqrt{1 + 4y^2 + 4z^2}}{1}\,dA$$

To finish up, we use the polar transformation $y = r\cos\theta$, $z = r\sin\theta$:

$$\int\int_R d\sigma = \int_0^{2\pi}\int_0^1 \sqrt{1 + 4r^2}\,r\,dr\,d\theta \qquad (\text{Let } u = 1 + 4r^2)$$

$$= \frac{1}{8}\int_0^{2\pi}\int_1^5 u^{1/2}\,du\,d\theta$$

$$= \frac{\pi}{4}\left(\frac{2}{3}(5^{3/2} - 1)\right) = \frac{\pi}{6}(5^{3/2} - 1)$$

(b)

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_{-\sqrt{x-z^2}}^{\sqrt{x-z^2}} dy\,dz\,dx.$$

(c) We minimize $f(x, y) = x^2 + y^2 - z/2$ on constraint $g(x, y, z) = -x + y^2 + z^2 = 0$. The method of Lagrange multipliers gives a system of nonlinear equations:

$$\begin{array}{llll} (1) & f_x = \lambda g_x & : & 2x = -\lambda \\ (2) & f_y = \lambda g_y & : & 2y = \lambda 2y \\ (3) & f_z = \lambda g_z & : & -1/2 = \lambda 2z \end{array}$$

From Equation (2) we have either $\lambda = 1$ or $y = 0$. Plugging $\lambda = 1$ into Equation (1) gives $x = -1/2$, which isn't on the constraint, so $\lambda \neq 1$. Using $y = 0$ in the constraint equation gives $x = z^2$, which, in conjuction with Equation (1), yields $\lambda = -2z^2$. Next, plugging $\lambda = -2z^2$ into Equation (3) and solving shows $z = 1/2$. Then $x = (1/2)^2 = 1/4$, and $\lambda = -2(1/4) = -1/2$. The minimum takes place at (1/4, 0 ,1/2) and has value

$$f(1/4, 0, 1/2) = (1/4)^2 + 0^2 - (1/2)/2 = 1/16 - 1/4 = -3/16.$$