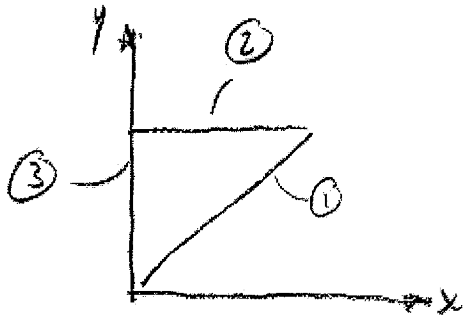
① $f = 20 - 4x - 16y + x^2 + 4y^2 = (x-2)^2 + 4(y-2)^2$



Crit. Pts $f_x = -4 + 2x = 0 \Rightarrow x = 2$

$f_y = -16 + 8y = 0 \Rightarrow y = 2$

$f(2,2) = 0$

Bndry ① $y = x \Rightarrow f = 20 - 20x + 5x^2$

$f_x = -20 + 10x \Rightarrow x = 2 = y = 2$

$f(2,2) = 0$

$f_{xx} = 10 \Rightarrow$ local min

Bndy ② $y = 4 \Rightarrow f = (x-2)^2 + 16$

$f_x = 2(x-2) = 0 \Rightarrow x = 2$

$f(2,4) = 16$

$f_{xx} = 2 \Rightarrow$ local min

Bndy ③ $x = 0$ $\quad f = 4 + 4(y-2)^2$

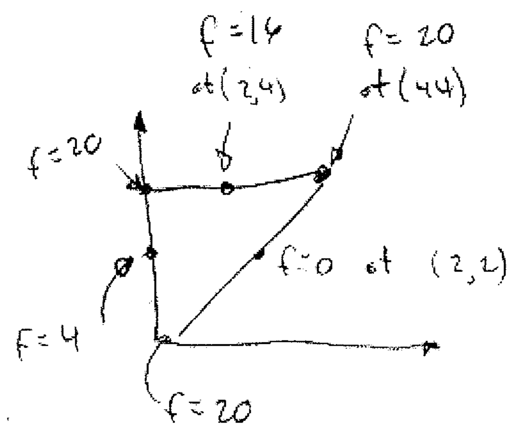
$f_y = 8(y-2) = 0 \Rightarrow y = 2$

$f(0,2) = 4$

$f_{yy} = 8 \Rightarrow$ local min

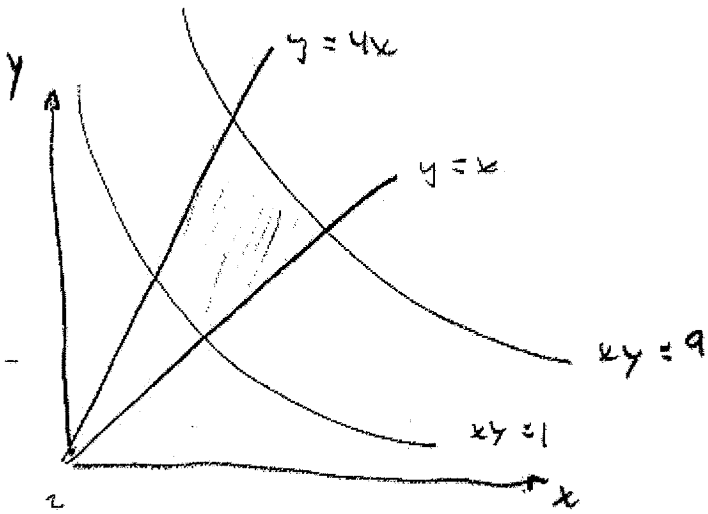
Corners

$(0,0)$	$(0,4)$	$(4,4)$
$f = 20$	$f = 20$	$f = 20$



[redacted]

(2) $I = \iint_R \left(\sqrt{\frac{y}{x}} + 3y \right) dx\, dy$



b) $x = \frac{u}{v}$ $\quad y = uv$

So Integrand terms are $\frac{y}{x} = \frac{uv}{u/v} = v^2$

$3y = 3uv$

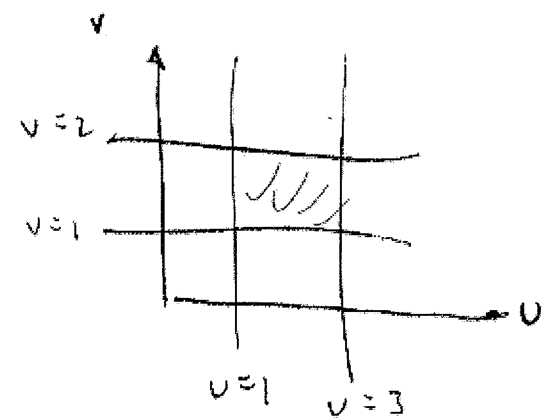
Boundaries are

$xy = 1 = \left(\frac{u}{v}\right)(uv) = u^2 \Rightarrow u = 1$

$xy = 9 = u^2 \Rightarrow u = 3$

$\frac{y}{x} = 1 = v^2 \Rightarrow v = 1$

$\frac{y}{x} = 4 = v^2 \Rightarrow v = 2$



c) $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ v & u \end{vmatrix}$

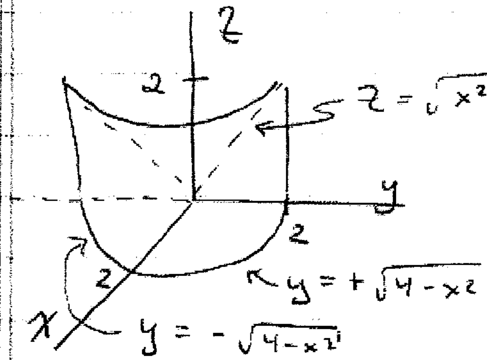
$= \frac{u}{v} + \frac{uv}{v^2} = 2\frac{u}{v}$

$\therefore I = \int_{v=1}^{2} \int_{u=1}^{3} \left(\sqrt{v^2} + 3uv \right) \left| 2\frac{u}{v} \right| du\, dv$

$= \int_{v=1}^{2} \int_{u=1}^{3} (2u + 6u^2)\, du\, dv = \int_{v=1}^{2} \left(u^2 + 2u^3 \Big|_1^3 \right) dv = \int_1^2 60\, dv$

$= 60$

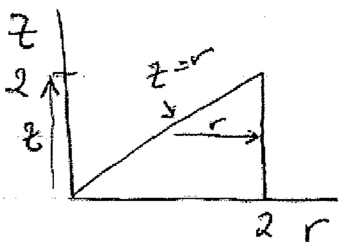
3 a $\displaystyle\int_{x=0}^{2}\int_{y=-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\int_{z=0}^{\sqrt{x^2+y^2}} dz\,dy\,dx$



$x^2+y^2=z^2=r^2=4$

$z=2$

b. θ goes from $-\pi/2$ to $\pi/2$, then in 2D

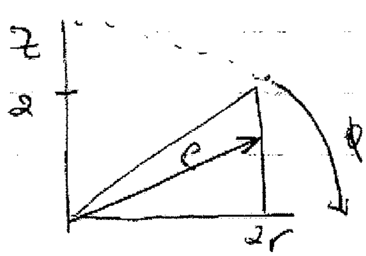


r goes from line $r=z$ to $r=2$

z goes from 0 to $z_{max}=2$

$$\int_{\theta=-\pi/2}^{\pi/2}\int_{z=0}^{2}\int_{r=z}^{2} r\,dr\,dz\,d\theta$$

c. θ is same as cylindrical, in 2D



ϕ goes from $\pi/4$ to max $\pi/2$

ρ goes from 0 to line

line is $r=2=\rho\sin\phi$

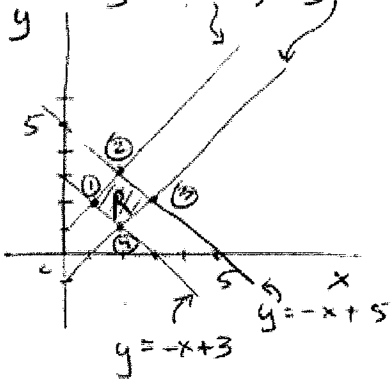
so $\rho=\dfrac{2}{\sin\phi}$ = line

$$\int_{\theta=-\pi/2}^{\pi/2}\int_{\phi=\pi/4}^{\pi/2}\int_{\rho=0}^{2/\sin\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

d. solve b $=\displaystyle\int_{z=0}^{2}\frac{r^2}{2}\Big|_{r=z}^{2}\ \theta\Big|_{-\pi/2}^{\pi/2}=\frac{\pi}{2}\int_{z=0}^{2}(4-z^2)\,dz$

$=\dfrac{\pi}{2}\left(4z-\dfrac{z^3}{3}\right)\Big|_{z=0}^{2}=\dfrac{\pi}{2}\left(8-\dfrac{8}{3}\right)=\dfrac{\pi}{2}\cdot 8\cdot\dfrac{2}{3}=\boxed{8\pi/3}$

4 a) $y = x+1$, $y = x-1$, $y = -x+5$, $y = -x+3$



b) $x = \frac{1}{2}(u+v)$ $\quad -\left(y = \frac{1}{2}(u-v)\right)$

$\underline{y = \frac{1}{2}(u-v)}$ $\quad \underline{x = \frac{1}{2}(u+v)}$

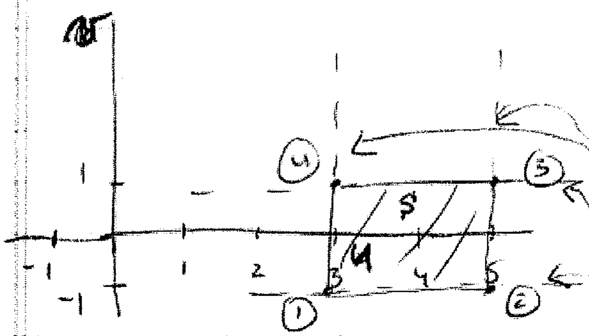
$x+y = u$ $\quad x - y = v$

(1) $x=1, y=2 \Rightarrow u = x+y = 1+2 = \boxed{3=u}$, $v = x-y = 1-2 = \boxed{-1=v}$

(2) $x=2, y=3 \Rightarrow u = 2+3 = \boxed{5=u}$, $v = 2-3 = \boxed{-1=v}$

(3) $x=3, y=2 \Rightarrow u = 2+3 = \boxed{5=u}$, $v = 3-2 = \boxed{1=v}$

(4) $x=2, y=1 \Rightarrow u = 2+1 = \boxed{3=u}$, $v = 2-1 = \boxed{1=v}$



equations

$y - x = 1 \Rightarrow x - y = \boxed{-1 = v}$

$y - x = 1 \Rightarrow x - y = \boxed{1 = v}$

$y + x = \boxed{5 = u}$

$y + x = \boxed{3 = u}$

c) $\iint_R e^{x+y}\,dx\,dy = \int_{u=3}^{5}\int_{v=-1}^{1} |J| e^{u}\,du\,dv$

3c cont.

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left|-\frac{1}{4} - \frac{1}{4}\right| = \frac{1}{2}$$

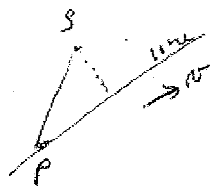
$$\int_{u=3}^{5}\int_{v=-1}^{1} \frac{1}{2} e^{u}\,du\,dv$$

$$= \frac{1}{2}\, v\Big|_{-1}^{1}\; e^{u}\Big|_{3}^{5} = \frac{1}{2}(2)(e^5 - e^3) = e^5 - e^3$$

5. $w = x^2 y^3 z^4$ $\quad x = 2t+1,\ y = 3t+2,\ z = 5t+4$

a) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt}$

$= (2xy^3z^4)(2) + (3x^2y^2z^4)(3) + (4x^2y^3z^3)(5)$

b) 

$d = \dfrac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$ $\quad \vec{v} = 2\hat{i} + 3\hat{j} + 5\hat{k}$

$|\vec{v}| = \sqrt{4+9+25}$

$= \sqrt{13+25}$

$= \sqrt{38}$

$\vec{PS}$ need a point P on line

choose $t = 0 \Rightarrow (1, 2, 4)$

$\vec{PS} = (1-0)\hat{i} + (2-0)\hat{j} + (4-0)\hat{k} = \hat{i} + 2\hat{j} + 4\hat{k}$

$d = |(\hat{i} * 2\hat{j} + 4\hat{k}) \times (2\hat{i} + 3\hat{j} + 5\hat{k})| / \sqrt{38}$

$\vec{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{vmatrix} = \hat{i}(10-12) - \hat{j}(5-8) + \hat{k}(3-4) = -2\hat{i} + 3\hat{j} - \hat{k}$

$|\vec{PS} \times \vec{v}| = \sqrt{4+9+1} = \sqrt{14}$

$\dfrac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \dfrac{\sqrt{14}}{\sqrt{38}} = \boxed{\dfrac{\sqrt{7}}{\sqrt{19}}}$

c) vector is $+2\hat{i} + 3\hat{j} + 5\hat{k} \Rightarrow -2x + 3y - z = c$

point is $(0,0,0) \Rightarrow 0 + 0 + 0 = c$

$\boxed{+2x + 3y + 5z = 0}$

5d directional derivative

$\nabla f|_{P_0} \cdot \hat{u} = (2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k})|_{P_0} \cdot \hat{u}$

$\hat{u} = \dfrac{2\hat{i} + 3\hat{j} + 5\hat{k}}{\sqrt{38}}$

$= (-2\hat{i} + 3\hat{j} - 4\hat{k}) \cdot (2\hat{i} + 3\hat{j} + 5\hat{k}) / \sqrt{38}$

$= (-4 + 9 - 20)/\sqrt{38}$

$= \boxed{-15/\sqrt{38}}$

6. $\vec{r}(t) = \cos t\,\hat{\imath} + \sqrt{2}\sin t\,\hat{\jmath} + \cos t\,\hat{k}$

Note: the point $(0,1,0)$ is not on the curve, but $(0,\sqrt{2},0)$ is (at $t=\pi/2$)

(a) $\hat{T} = \frac{\vec{v}}{|\vec{v}|}$ $\qquad \vec{v} = -\sin t\,\hat{\imath} + \sqrt{2}\cos t\,\hat{\jmath} - \sin t\,\hat{k}$

$|\vec{v}| = \sqrt{\sin^2 t + 2\cos^2 t + \sin^2 t} = \sqrt{2(\sin^2 t + \cos^2 t)} = \sqrt{2}$

$\Rightarrow \boxed{\hat{T} = -\frac{1}{\sqrt{2}}\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} - \frac{1}{\sqrt{2}}\sin t\,\hat{k}} \Rightarrow \boxed{\hat{T}(\pi/2) = -\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{k}}$

(b) $\hat{N} = \frac{d\hat{T}}{dt} \Big/ \left|\frac{d\hat{T}}{dt}\right|$ $\qquad \frac{d\hat{T}}{dt} = -\frac{1}{\sqrt{2}}\cos t\,\hat{\imath} - \sin t\,\hat{\jmath} - \frac{1}{\sqrt{2}}\cos t\,\hat{k}$

$\left|\frac{d\hat{T}}{dt}\right| = \sqrt{\tfrac{1}{2}\cos^2 t + \sin^2 t + \tfrac{1}{2}\cos^2 t} = \sqrt{1} = 1$

$\Rightarrow \boxed{\hat{N} = -\left[\frac{1}{\sqrt{2}}\cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + \frac{1}{\sqrt{2}}\cos t\,\hat{k}\right]} \Rightarrow \boxed{\hat{N}(\pi/2) = -\hat{\jmath}}$

(c) $\kappa = \left|\frac{d\hat{T}}{ds}\right| = \left|\frac{d\hat{T}}{dt}\frac{dt}{ds}\right| = \left|\frac{d\hat{T}}{dt}\right|\frac{1}{|\vec{v}|}$

$= (1)\left(\frac{1}{\sqrt{2}}\right)$ from above

$\Rightarrow \boxed{\kappa = \frac{1}{\sqrt{2}}}$

(d) Length of $C = \int_C ds = \int_0^{2\pi} |\vec{v}|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = \boxed{2\sqrt{2}\,\pi}$

7. Maximize $T = 900xyz^2$ subject to constraint $g(x,y,z) = x^2+y^2+z^2-1 = 0$

Lagrange multipliers: $\nabla T = \lambda \nabla g$

$\Rightarrow 900yz^2\,\hat{\imath} + 900xz^2\,\hat{\jmath} + 1800xyz\,\hat{k} = \lambda(2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k})$

$$\Rightarrow \begin{cases} 450yz^2 = \lambda x & (1) \\ 450xz^2 = \lambda y & (2) \\ 900xyz = \lambda z & (3) \\ x^2+y^2+z^2 = 1 & (4) \end{cases}$$

(3) $\Rightarrow 900xy = \lambda$ OR $z = 0$

If $x=0$, $y=0$ or $z=0$, $T=0$ which is clearly not the maximum, so ignore these cases

$\lambda = 900xy \overset{(1)}{\Rightarrow} 450yz^2 = 900x^2y \Rightarrow z^2 = 2x^2$ (or $y=0$)

$\Rightarrow z = \pm\sqrt{2}\,x$

$\overset{(2)}{\Rightarrow} 450xz^2 = 900xy^2 \Rightarrow z^2 = 2y^2$ (or $x=0$)

$\Rightarrow z = \pm\sqrt{2}\,y$

(4) $\Rightarrow x^2+y^2+z^2 = \tfrac{1}{2}z^2 + \tfrac{1}{2}z^2 + z^2 = 2z^2 = 1 \Rightarrow z = \pm 1/\sqrt{2}$

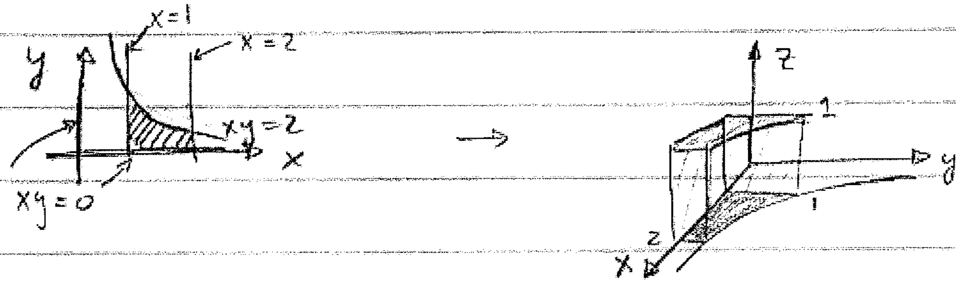
$\Rightarrow x = \left(\pm\frac{1}{\sqrt{2}}\right)\left(\pm\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{2}$ & $y = \left(\pm\frac{1}{\sqrt{2}}\right)\left(\pm\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{2}$ $\Rightarrow$ 8 critical points

$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right) \Rightarrow T = 900\left(\pm\tfrac{1}{2}\right)\left(\pm\tfrac{1}{2}\right)\left(\pm\tfrac{1}{\sqrt{2}}\right)^2 = \pm 900\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \pm\frac{900}{8}$

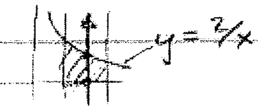
So max temp is $\boxed{\frac{900}{8} = 112.5}$ at $\boxed{(x,y,z) = \left(\tfrac{1}{2}, \tfrac{1}{2}, \pm\tfrac{1}{\sqrt{2}}\right) \text{ \& } \left(-\tfrac{1}{2}, -\tfrac{1}{2}, \pm\tfrac{1}{\sqrt{2}}\right)}$

Max temp is $112.5 < 150 \Rightarrow$ $\boxed{\text{Ball DOES NOT explode}}$

8 (a)



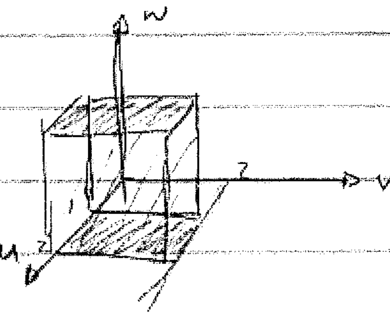
(b) $M = \iiint_D \delta(x,y,z)\, dV$



$= \int_0^1 \int_1^2 \int_0^{2/x} (x^2 y + 3xyz)\, dy\, dx\, dz$

(c) Boundaries:

$$\left.\begin{aligned} y &= 0 \\ xy &= 2 \\ x &= 1 \\ x &= 2 \\ z &= 0 \\ z &= 1 \end{aligned}\right\} \Leftarrow \begin{pmatrix} u = x \\ v = xy \\ w = 3z \end{pmatrix} \Rightarrow \left\{\begin{aligned} v &= 0 \\ v &= 2 \\ u &= 1 \\ u &= 2 \\ w &= 0 \\ w &= 3 \end{aligned}\right.$$



(d) $M = \int_0^3 \int_0^2 \int_1^2 (\underbrace{uv}_{x^2y} + \underbrace{vw}_{3xyz}) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$

$x = u$
$y = v/x = v/u$
$w = z/3$

$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & 1/3 \end{vmatrix} = 1 \begin{vmatrix} \frac{1}{u} & 0 \\ 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}$

$\Rightarrow M = \int_0^3 \int_0^2 \int_1^2 \frac{uv + vw}{3u}\, du\, dv\, dw = \int_0^3 \int_0^2 \int_1^2 \frac{v}{3} + \frac{vw}{3u}\, du\, dv\, dw$

$= \int_0^3 \int_0^2 \left[\frac{v}{3} u + \frac{vw}{3} \log(u) \right]_1^2 dv\, dw = \int_0^3 \int_0^2 \frac{v}{3} + \frac{vw}{3} \log(2)\, dv\, dw$

$= \int_0^3 \left[\frac{1}{3}(1 + \log(2) w) \frac{v^2}{2} \right]_0^2 dw = \int_0^3 \frac{2}{3}(1 + \log(2) w)\, dw$

$= \frac{2}{3}[w]_0^3 + \frac{\log(2)}{3} [w^2]_0^3 = \frac{2}{3}(3) + \frac{9\log(2)}{3}$

$= \boxed{2 + 3\log(2)}$

9) rectangular box, no lid, 12m of cardboard

$f(x,y,z) = V = xyz$

$g(x,y,z) = 2xz + 2yz + xy = 12$ (4) (surface area)

$\nabla f = \lambda \nabla g$ gives:

$$\begin{aligned} yz &= \lambda(2z + y) \\ xz &= \lambda(2z + x) \\ xy &= \lambda(2x + 2y) \end{aligned} \quad \overset{\text{trick}}{\Rightarrow} \quad \begin{aligned} xyz &= \lambda(2zx + yx) \quad (1) \\ xyz &= \lambda(2zy + xy) \quad (2) \\ xyz &= \lambda(2xz + 2yz) \quad (3) \end{aligned}$$

$\lambda \neq 0$, $x, y, z \neq 0$

From (1) and (2) we have $2zx + \cancel{xy} = 2zy + \cancel{xy} \Rightarrow x = y$

From (2) and (3) we now have $\cancel{2zy} + xy = 2xz + \cancel{2zy} \Rightarrow xy = 2xz$

$\Rightarrow y = 2z$

Now substitute $x = y = 2z$ into (4) to get:

$4z^2 + 4z^2 + 4z^2 = 12 \Rightarrow z = 1, x = 2, y = 2$

$=$ max Volume $= 4$