Exam 1 Solutions

① a) $\underline{A} \circ (\underline{B} \times \underline{C}) \overset{?}{=} (\underline{A} \times \underline{B}) \circ \underline{C}$
$\overset{?}{=} \underline{C} \circ (\underline{A} \times \underline{B})$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \overset{?}{=} \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

so Always true

b) $\underline{A} \times \underline{B} = \underline{C} \times \underline{B}$ and $\underline{B} \neq \underline{0}$

$(\underline{A} - \underline{C}) \times \underline{B} = \underline{0}$

So $(\underline{A} - \underline{C})$ is $\parallel$ to $\underline{B}$, but not necessarily $(\underline{A} - \underline{C}) = \underline{0}$.

So Not Always true.

c) $\underbrace{(\underline{A} \times \underline{B}) \circ \underline{A}}_{=\underline{0}} \overset{?}{=} \underbrace{(\underline{B} \times \underline{A}) \circ \underline{B}}_{=\underline{0}}$ so Always true

d) If $|\underline{V}|$ is not a constant, then $\underline{V}$ and $\underline{a}$ may not be $\perp$.

So Not Always true.

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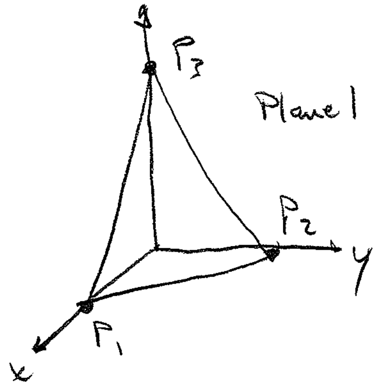
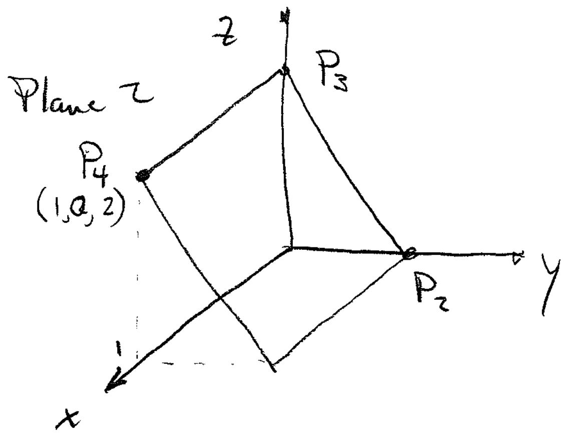




a) For plane 1 $\quad \underline{P_1P_2} = -\hat{i} + \hat{j} + 0\hat{k}$

$\underline{P_1P_3} = -\hat{i} + 0\hat{j} + 2\hat{k}$

Normal to plane 1 is then $\underline{n}_1 = \underline{P_1P_2} \times \underline{P_1P_3}$

$= \dots$

$= 2\hat{i} + 2\hat{j} + \hat{k}$

Thus standard eqn of plane 1 is $\quad 2x + 2y + z = D$

Use $P_1(1,0,0)$ in $\quad 2x + 2y + z = 2\cdot1 + 2\cdot0 + 0 = D = 2$

Thus $\quad 2x + 2y + z = 2$

$$\hat{n}_1 = \frac{\underline{n}_1}{|\underline{n}_1|} = \frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$$

b) For plane 2 $\quad \underline{P_2P_3} = 0\hat{i} + -\hat{j} + 2\hat{k}$

$\underline{P_3P_4} = \hat{i} + 0\hat{j} + 0\hat{k}$

so $\quad \underline{n}_2 = \underline{P_2P_3} \times \underline{P_3P_4} = \dots = 0\hat{i} + 2\hat{j} + \hat{k}$

Std eqn is then $\quad 2y + z = D$

Plug in $P_4(1,0,2)$ to get $\quad 2\cdot0 + 2 = D = 2$

$\therefore \quad 2y + z = 2$

$$\hat{n}_2 = \frac{\underline{n}_2}{|\underline{n}_2|} = \frac{1}{\sqrt{5}}(0\hat{i} + 2\hat{j} + \hat{k})$$

c) $\quad \underline{n}_1 \circ \underline{n}_2 = |\underline{n}_1||\underline{n}_2| \cos\theta$

so

$$\cos\theta = \frac{\underline{n}_1 \circ \underline{n}_2}{|\underline{n}_1||\underline{n}_2|} = \frac{(2\hat{i} + 2\hat{j} + \hat{k})\circ(0\hat{i} + 2\hat{j} + \hat{k})}{3\sqrt{5}}$$

$$= \frac{5}{3\sqrt{5}} = \sqrt{5}/3$$

(2)

d) A line through the origin in the direction of $\underset{\sim}{n}_1$ is

$$\underset{\sim}{r}(t) = \underset{\sim}{0} + (t)\underset{\sim}{n}_1$$
$$= 2t\,\hat{i} + 2t\,\hat{j} + t\,\hat{k}$$

We want to find the value t^* for which $\underset{\sim}{r}(t^*)$ is on plane 2. Thus since plane 2 is defined by $2y + z = 2$

$$2(2t^*) + (t^*) = 2$$
$$5t^* = 2 \quad \text{or} \quad t^* = 2/5$$

Thus $\underset{\sim}{r}(t^*) = 2\left(\frac{2}{5}\right)\hat{i} + 2\left(\frac{2}{5}\right)\hat{j} + \left(\frac{2}{5}\right)\tilde{k}$

$$= 4/5\,\hat{i} + 4/5\,\hat{j} + 2/5\,\hat{k}$$

so the intersection point is $(4/5, 4/5, 2/5)$. ←

(3) $\underline{r}(t^*) = 0\hat{i} + 3\hat{j} + 4\hat{k}$ $\qquad |\underline{v}| = ¢$

$\underline{v}(t^*) = 5\hat{i} + 5\hat{j} + 0\hat{k}$

$\underline{a}(t^*) = 0\hat{i} + 0\hat{j} + 7\hat{k}$

a) $\hat{T} = \frac{\underline{v}}{|\underline{v}|} = \frac{1}{\sqrt{50}}(5\hat{i} + 5\hat{j} + 0\hat{k}) = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j} + 0\hat{k})$

b) $\underline{a} = \kappa|\underline{v}|^2 \hat{N} + \cancel{\frac{d|\underline{v}|}{dt}}^{\,0} \hat{T}$

so $\hat{N} = \frac{\underline{a}}{|\underline{a}|} = \frac{1}{7}(0\hat{i} + 0\hat{j} + 7\hat{k}) = (0\hat{i} + 0\hat{j} + \hat{k})$

c) From $\underline{a} = \kappa|\underline{v}|^2 \hat{N}$ we see that $\kappa|\underline{v}|^2 = |\underline{a}|$

so $\kappa = \frac{|\underline{a}|}{|\underline{v}|^2} = \frac{7}{(\sqrt{50})^2} = 7/50$

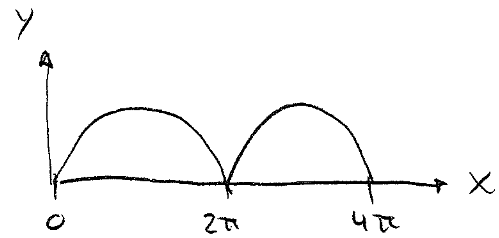
Note: could also use $\kappa = \frac{|\underline{v} \times \underline{a}|}{|\underline{v}|^3}$

d) Insufficient info.

e) Since $|\underline{v}| = ¢$ then $\underline{v}$ is $\perp$ $\underline{a}$ for all t, in particular at $t = 2t^*$. Thus $\underline{v} \circ \underline{a} = 0$ always.

(4) $\underline{r}(t) = (t - \sin t)\,\hat{i} + (1 - \cos t)\,\hat{j}$

$\underline{V}(t) = (1 - \cos t)\,\hat{i} + \sin t\,\hat{j}$



a) height $= 1 - \cos t$

$= 0$ if $t = 0, 2\pi, 4\pi, \dots$

So 2 cycles (rotations) means $t = 4\pi$

b) S = arc length $= \displaystyle\int_{t=0}^{4\pi} |\underline{V}|\,dt$

$$= \int_{t=0}^{4\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\,dt$$

$$|\underline{V}| = \sqrt{(1-\cos t)^2 + (\sin t)^2}$$
$$= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$$
$$= \sqrt{2 - 2\cos t}$$
$$= \sqrt{2}\sqrt{1 - \cos t}$$

But using double ∡

$$\sin^2(t/2) = \frac{1 - \cos t}{2}$$
$$|\underline{V}| = \sqrt{2}\sqrt{2\sin^2(t/2)}$$
$$= 2\left|\sin \frac{t}{2}\right|$$

$$= \int_{t=0}^{4\pi} 2\left|\sin\frac{t}{2}\right|\,dt$$

$$= 2 \cdot 2\int_{t=0}^{2\pi} \sin\frac{t}{2}\,dt$$

$$= 4\left[-2\cos\frac{t}{2}\right|_{t=0}^{2\pi}$$

$$= -8(\cos\pi - \cos 0)$$

$$= 16 \quad \leftarrow$$

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a) $z^2 + y^2 + x = 0$ $A = E = F = G = 0$

$B = C = D = 1$

(The negative version is also ok w $D = -1$ & $B = C = -1$.)

b) $x^2 - y^2 + z^2 = 0$ $D, E, F, G = 0$

$A = C = 1$ $B = -1$

(Negative version is also ok.)

c) $x^2 - y^2 - z^2 = 1$

$A = G = 1$

$B = C = -1$

$D, E, F = 0$

d) $x^2 - y^2 + z^2 = 1$

$A = G = 1$

$B = C = -1$

$D = E = F = 0$