INSTRUCTIONS: Electronic devices, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

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1. (20 points) You are to locate potential sites for forest ranger patrol cabins in a section of forest where the height of the surface can be described by the function $f(x, y) = xy - \frac{x^3}{3} - \frac{y^3}{3}$. The main requirement is that the ground be level where the cabins are to be located.

   (a) Determine the location of all potential cabin sites.

   (b) Classify these potential cabin sites (hill top, valley bottom, etc.) and determine the value of $f(x, y)$ at each of these locations.

   (c) If you were to build a network of roads directly connecting each cabin site to another, what would be the steepest slope you would encounter when driving along these roads? Hint: think carefully about what you are trying to maximize here.

2. (20 points) On a given day, the temperature distribution (in degrees Fahrenheit) is given by $T(x, y) = 50 + xy + y^2$. Two cruise ships leave a port located at $(0, 0)$. Ship A travels along the path defined by $y = x^2$ and ship B travels along the path $y = \sqrt{x}$. The ships travel at the same speed of 10 knots.

   (a) When the ships meet, at what rate is the temperature $T(x, y)$ changing with respect to *distance* for each ship?

   (b) If the ship experiencing the largest rate of increase in temperature with respect to *distance* travels for a short time $\Delta t = 0.1$, by approximately how much will the temperature change for the passengers on the ship?

   (c) If the other ship travels for a short distance $\Delta s = 0.1$, by approximately how much will the temperature change for the passengers on the ship?

3. (20 points) The strength of a cell phone signal in a certain section of a city can be described by the function $f(x, y) = 9 - \frac{x^3}{3} - \frac{y^3}{3}$ where $x \geq 0$ and $y \geq 0$. A highway that passes through the city can be described by xy = 4. As you drive along the highway, at what location do you get the strongest cell phone signal?

4. (20 points) Suppose you are to evaluate the function $f(x, y, z) = x^2y^3/z$, however you put in the wrong values for each variable. In particular, the value you use for each variable is 10% high. Estimate the relative error on the calculated value of f.

5. (20 points) Consider the function $f(x, y) = x^3 + y^3 + x^2y^2$.

(a) Calculate the *first order* Taylor approximation to $f(x, y)$ near the point $(1, 1)$.

(b) Use your result from part (a) to estimate the value of $f(1.1, 1.1)$.

(c) Now suppose you actually worked out the *second order* Taylor approximation to $f(x, y)$ near the point $(1, 1)$ (you don't actually need to do it). Calculate an "upper bound on the error" associated with this *second order* approximation assuming that you only use values of x and y such that $|x - 1| \leq 0.1$ and $|y - 1| \leq 0.1$.

Projections and distances

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|1 + (f'(x))^2|^{3/2}} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{|\dot{x}^2 + \dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f) \cdot \mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda \nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$\begin{aligned} f(x, y) = \; & f(x_0, y_0) + \Big[(x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0)\Big] \\ & + \frac{1}{2!}\Big[(x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0)f_{xy}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0)\Big] \\ & + \frac{1}{3!}\Big[(x - x_0)^3 f_{xxx}(x_0, y_0) + 3(x - x_0)^2(y - y_0)f_{xxy}(x_0, y_0) \\ & \quad + 3(x - x_0)(y - y_0)^2 f_{xyy}(x_0, y_0) + (y - y_0)^3 f_{yyy}(x_0, y_0)\Big] + \cdots \end{aligned}$$

Linear approximation error

$$|E(x, y)| \leq \frac{1}{2}M(|x - x_0| + |y - y_0|)^2, \qquad \text{where } \max\{\,|f_{xx}|,\, |f_{xy}|,\, |f_{yy}|\,\} \leq M$$