INSTRUCTIONS: Books, notes, crib sheets, and electronic devices are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show and explain your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (25 points) An old, dusty grain storage facility was built with a floor described by the region in the first quadrant of the xy-plane enclosed by the x-axis, the line $y = x$, and the arcs $r = 1$ and $r = 2$. The roof of the building has height $z = \sqrt{4 - x^2 - y^2}$. The density of dust particles in the air (particles per unit volume) is given by $\delta = \sqrt{4 - x^2 - y^2}$.
   (a) Calculate the area of the floor. (Use calculus techniques! However, feel free to check your answer with geometry.)
   (b) Calculate the volume enclosed by the building.
   (c) Calculate the number of dust particles in the building.

2. (25 points) Consider the integral
$$I = \iint_{R_{xy}} 8xy\ dx\, dy\,,$$
   where R_{xy} is the region in the xy-plane bounded by the curves $x = 0$, $y = x$, $y = 1 - x$, and $y = 2 - x$.
   (a) The substitution $u = x + y$ and $v = x - y$ will simplify the evaluation of I. Find x and y in terms of u and v using the given substitution. Be sure to check this because the rest of the problem depends on this result!
   (b) Transform the original region R_{xy} into its corresponding region R_{uv} in the uv-plane. Make two clear sketches, one of the original region of integration R_{xy} in the xy-plane, and one of the new region of integration R_{uv} in the uv-plane. Be sure to label all axes, boundaries, intersection points, etc. on each sketch.
   (c) Rewrite the integral for I over the region R_{uv} in the uv-plane in terms of u and v.
   (d) Evaluate I in terms of u and v.

3. (25 points) The integral
$$V = \int_{\theta=0}^{2\pi} \int_{r=R}^{\sqrt{3}R} \int_{z=R}^{\sqrt{4R^2-r^2}} r\ dz\ dr\ d\theta$$
   calculates the volume of an object.
   (a) Make a clear sketch of a cross-section of the object in a rz-plane (this is a constant θ plane in cylindrical coordinates) clearly labeling the bounding surfaces of the region of integration. (If you have trouble with this, you may "buy" a sketch of the shape of the region of integration for 5 points. **This sketch will only show the shape of the region**, so you will still need to supply the remaining details. The offer to buy this sketch ends at 6:15 PM!)
   (b) Express V in cylindrical coordinates using the order $dr\, dz\, d\theta$.
   (c) Express V in spherical coordinates using the order $d\rho\, d\phi\, d\theta$.
   (d) Express V in spherical coordinates using the order $d\phi\, d\rho\, d\theta$.
   (e) Evaluate one of the integrals above to determine the value of V.

4. (25 points) Consider a counter-clockwise path along the boundary of the region in the first quadrant inside the curve $x^2 + y^2 = 1$, and the vector function given by $\mathbf{F} = xy\,\mathbf{i} + xy\,\mathbf{j}$.
   (a) Sketch the path and give parametrizations for each section of the path.
   (b) Calculate the total **flow** (circulation) along path C.
   (c) Calculate the total outward **flux** along path C.

Projections and distances $\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A}$ $d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$ $d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Polar coordinates $x = r\cos\theta$ $y = r\sin\theta$ $r^2 = x^2 + y^2$ $dA = dx\,dy = r\,dr\,d\theta$

Cylindrical and spherical coordinates

Cylindrical to Rectangular	Spherical to Cylindrical	Spherical to Rectangular
$x = r\cos\theta$	$r = \rho\sin\phi$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$z = \rho\cos\phi$	$y = \rho\sin\phi\sin\theta$
$z = z$	$\theta = \theta$	$z = \rho\cos\phi$

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Substitutions in multiple integrals

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\left(x(u,v),y(u,v)\right)\,|J(u,v)|\,du\,dv \quad \text{where} \quad J(u,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Mass, moments, and center of mass Mass $M = \iint_R \delta\,dA$

Moments $M_x = \iint_R y\,\delta\,dA$ $M_y = \iint_R x\,\delta\,dA$ Center of mass $\bar{x} = M_y/M$ $\bar{y} = M_x/M$

Flow and flux Flow $= \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot\mathbf{V}\,dt = \int_C \mathbf{F}\cdot d\mathbf{r}$ Flux $= \int_C \mathbf{F}\cdot\mathbf{n}\,ds$