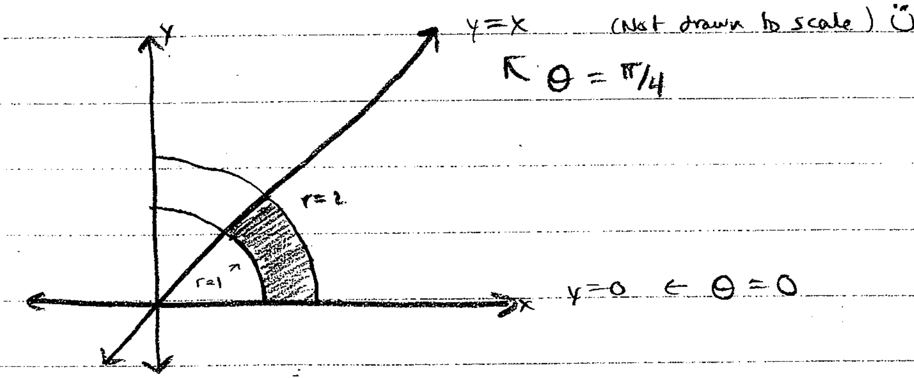
1



ⓐ $\int_0^{\pi/4}\int_1^2 r\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^2}{2}\right]_1^2 d\theta = \frac{1}{2}\int_0^{\pi/4}[4-1]\,d\theta$

$= \frac{3}{2}\int_0^{\pi/4} d\theta = \frac{3}{2}[\theta]_0^{\pi/4} = \frac{3}{2}\left[\frac{\pi}{4}-0\right]$

$= \boxed{3\pi/8 \text{ units}^2}$

8

ⓑ $z = \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$

Bounds = $z=0,\ z=\sqrt{4-r^2}$

Volume $= \int_0^{\pi/4}\int_1^2\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$ $\qquad u=(4-r^2)^{3/2}$

$= \int_0^{\pi/4}\int_1^2\left[\sqrt{4-r^2}\right] r\,dr\,d\theta$ $\qquad du = \frac{3}{2}(4-r^2)^{1/2}(-2r)$

$= \int_0^{\pi/4}\left[-\frac{1}{3}(4-r^2)^{3/2}\right]_1^2 d\theta$ $\qquad du = -3r(4-r^2)^{1/2}$

$= -\frac{1}{3}\int_0^{\pi/4}\left[0-(3)^{3/2}\right]d\theta$ $\qquad -\frac{du}{3} = r(4-r^2)^{1/2}$

$= \frac{1}{3}\int_0^{\pi/4}[3\sqrt{3}]\,d\theta = \sqrt{3}\int_0^{\pi/4} d\theta$ $\qquad \to -\frac{1}{3}$ needed

$= \sqrt{3}\,[\theta]_0^{\pi/4} = \boxed{\frac{\sqrt{3}}{4}\pi \text{ units}^3}$ 9

ⓒ # of particles $= \int_0^{\pi/4}\int_1^2\int_0^{\sqrt{4-r^2}} \delta(x,y)\, r\,dz\,dr\,d\theta$

$\delta(r) = \sqrt{4-r^2}$ b/c $\delta(x,y) = \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$

# of particles $= \int_0^{\pi/4}\int_1^2\int_0^{\sqrt{4-r^2}} \sqrt{4-r^2}\, r\,dz\,dr\,d\theta$

$= \int_0^{\pi/4}\int_1^2 (4-r^2)\,r\,dr\,d\theta$

$= \int_0^{\pi/4}\int_1^2 (4r-r^3)\,dr\,d\theta$

$= \int_0^{\pi/4}\left[2r^2 - \frac{r^4}{4}\right]_1^2 d\theta$ 8

$= \int_0^{\pi/4}\left[2(4)-4-2+\frac{1}{4}\right]d\theta$

$= \int_0^{\pi/4}\left[\frac{9}{4}\right]d\theta = \frac{9}{4}[\theta]_0^{\pi/4} = \boxed{9\pi/16 \text{ particles}}$

② $I = \iint 8xy\,dx\,dy$, $x=0$, $y=x$, $y=1-x$, $y=2-x$

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ⓐ $u = x+y$, $v = x-y$

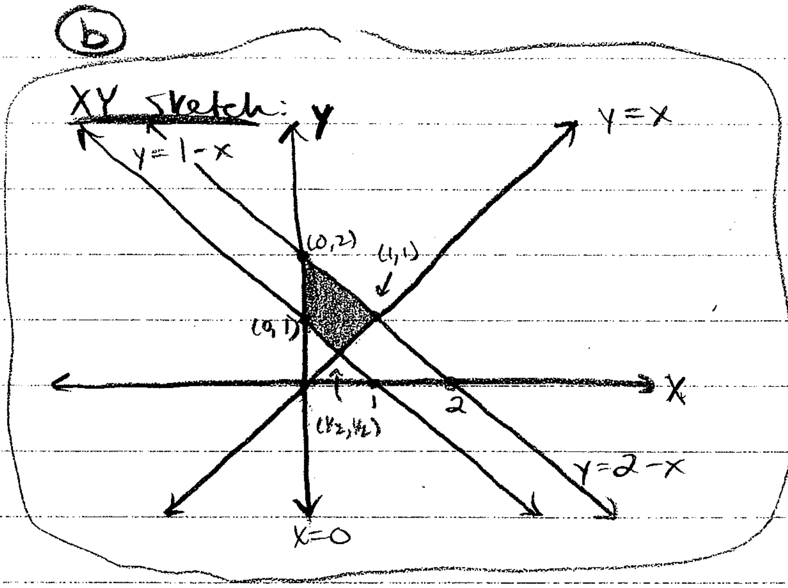
$x = u - y \Longrightarrow v = u - y - y \qquad v = u - 2y \qquad v - u = -2y$

$\boxed{y = \frac{u-v}{2}}$

$x = u - \left(\frac{u-v}{2}\right) = \frac{2u}{2} - \frac{u}{2} + \frac{v}{2}$

$= \frac{u}{2} + \frac{v}{2} = \frac{u+v}{2} \Rightarrow \boxed{x = \frac{u+v}{2}}$

ⓑ

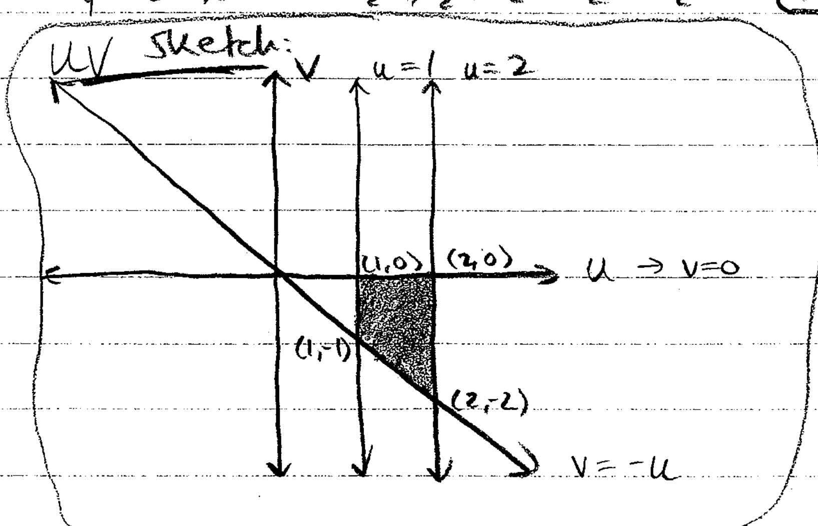


$x=0 \Rightarrow \frac{u}{2} + \frac{v}{2} = 0 \qquad u+v=0 \qquad u=-v$ or $\boxed{v=-u}$

$y=x \Rightarrow \cancel{\frac{u}{2}} - \frac{v}{2} = \cancel{\frac{u}{2}} + \frac{v}{2} \qquad -v = v \Rightarrow \boxed{v=0}$

$y=1-x \Rightarrow \frac{u}{2} - \cancel{\frac{v}{2}} = 1 - \frac{u}{2} - \cancel{\frac{v}{2}} \Rightarrow \boxed{u=1}$

$y=2-x \Rightarrow \frac{u}{2} - \cancel{\frac{v}{2}} = 2 - \frac{u}{2} - \cancel{\frac{v}{2}} \Rightarrow \boxed{u=2}$



(c) $J(u,v) = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/4 - 1/4 = -1/2$

$$\int_1^2 \int_0^{-u} -\tfrac{1}{2}(8)\left(\frac{u+v}{2}\right)\left(\frac{u-v}{2}\right) dv\,du$$

$$= -\int_1^2 \int_0^{-u} (u+v)(u-v)\,dv\,du$$

$$= -\int_1^2 \int_0^{-u} (u^2+uv-uv-v^2)\,dv\,du$$

$$= -\int_1^2 \int_0^{-u} (u^2-v^2)\,dv\,du$$

$$= \boxed{\int_1^2 \int_0^{-u} (v^2-u^2)\,dv\,du}$$

(d) $\int_1^2 \left[\frac{v^3}{3} - vu^2\right]_0^{-u} du = \int_1^2 \left[\frac{-u^3}{3} + u^3 - 0 - 0\right] du$

$$= \int_1^2 \left[2/3\, u^3\right] du = 2/3 \left[\frac{u^4}{4}\right]_1^2$$

$$= \frac{2}{3}\left[\frac{1}{4}\right](16-1) = \frac{1\,\cancel{2}\cdot 1\cdot \cancel{15}^{\,5}}{\cancel{3}\cdot \cancel{4}_{\,2}} = \boxed{5/2}$$

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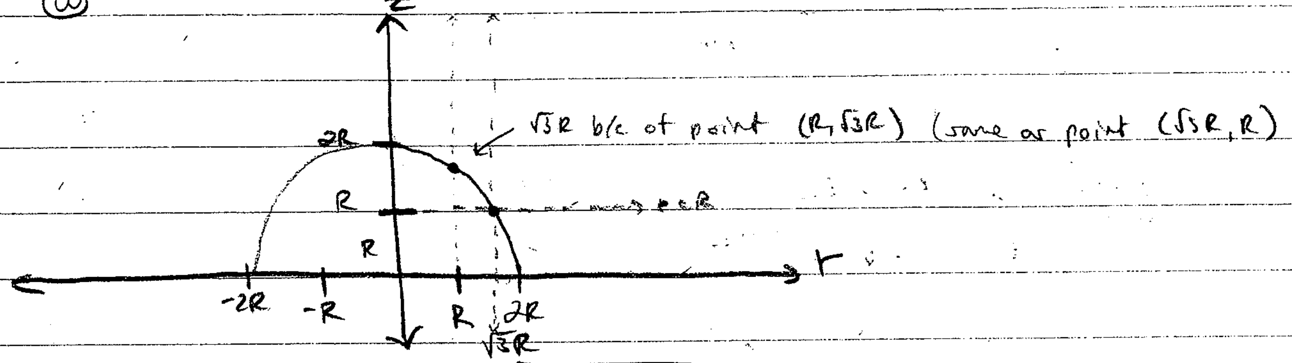
(3) $V = \int_0^{2\pi} \int_R^{\sqrt{3}R} \int_R^{\sqrt{4R^2 - r^2}} r\,dz\,dr\,d\theta$

Since $\theta = 0 \Rightarrow \theta = 2\pi$ is held constant:

(a)



$z = R$ to $z = \sqrt{4R^2 - r^2}$

$z^2 = 4R^2 - r^2$

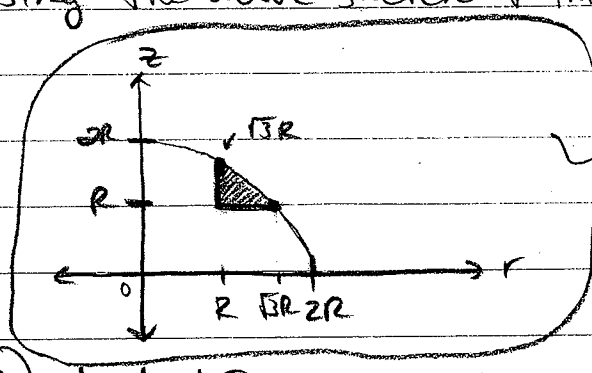
$z^2 = 4R^2 - x^2 - y^2$

$z^2 + x^2 + y^2 = 4R^2 \rightarrow$ Sphere Radius $= 2R$

At $r = 0$, $z = 2R$

$r = R$, $r = \sqrt{3}R$

Using the above sketch + information:



5/5

$z = \sqrt{4R^2 - r^2}$

$z^2 = 4R^2 - r^2$

$r^2 = 4R^2 - z^2$

$r = \oplus\sqrt{4R^2 - z^2}$

(b) $dr\,dz\,d\theta \Rightarrow$

$\int_0^{2\pi} \int_R^{\sqrt{3}R} \int_R^{\sqrt{4R^2 - z^2}} r\,dr\,dz\,d\theta$

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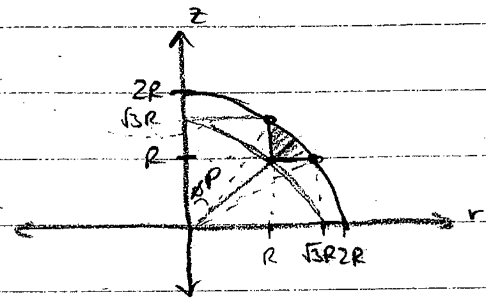
(c)



$d\rho\, d\phi\, d\theta$

$\theta \Rightarrow 0,\ \theta \Rightarrow 2\pi$

$\phi \Rightarrow \sin\phi = \frac{\sqrt{3}R}{\rho} \quad \cos\phi = \frac{R}{\rho}$

$\tan\phi = \sqrt{3} \Rightarrow \pi/6$

$\phi \Rightarrow \sin\phi = \frac{R}{\rho} \quad \cos\phi = \frac{\sqrt{3}}{\rho}$

$\tan\phi = \frac{1}{\sqrt{3}} \Rightarrow \pi/3$

Two Integrals: Divided by $\phi = \pi/4$

Both ρ leave at $2R$

1st: $r = \rho\sin\phi = R$

$\rho = R\csc\phi$ from $\pi/6$ to $\pi/4$

$\int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_{R\csc\phi}^{2R} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

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2nd: $z = \rho\cos\phi = R \Rightarrow \rho = R\sec\phi$ from $\pi/4$ to $\pi/3$

$\int_0^{2\pi}\int_{\pi/4}^{\pi/3}\int_{R\sec\phi}^{2R} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$\therefore \Rightarrow$

$$\boxed{\int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_{R\csc\phi}^{2R} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta + \int_0^{2\pi}\int_{\pi/4}^{\pi/3}\int_{R\sec\phi}^{2R} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

(d) 1st: $r = \rho\sin\phi = R \quad \phi = \sin^{-1}\left(\frac{R}{\rho}\right)$

from $\rho = \sqrt{2}R$ to $\rho = 2R$

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1st: $\int_0^{2\pi}\int_{\sqrt{2}R}^{2R}\int_{\sin^{-1}(\frac{R}{\rho})}^{\pi/4} \rho^2\sin\phi\, d\phi\, d\rho\, d\theta$

2nd: $z = \rho\cos\phi = R \quad \phi = \cos^{-1}\left(\frac{R}{\rho}\right)$

from $\rho = \sqrt{2}R$ to $\rho = 2R$

2nd: $\int_0^{2\pi}\int_{\sqrt{2}R}^{2R}\int_{\pi/4}^{\cos^{-1}(\frac{R}{\rho})} \rho^2\sin\phi\, d\phi\, d\rho\, d\theta$

$$\therefore \Rightarrow \boxed{\int_0^{2\pi}\int_{\sqrt{2}R}^{2R}\int_{\sin^{-1}(R/\rho)}^{\pi/4} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi}\int_{\sqrt{2}R}^{2R}\int_{\pi/4}^{\cos^{-1}(R/\rho)} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta}$$

(e) From (b);

$$\int_0^{2\pi}\int_R^{\sqrt{3}R}\int_R^{\sqrt{4R^2-z^2}} r\,dr\,dz\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_R^{\sqrt{3}R}\left[\left(\sqrt{4R^2-z^2}\right)^2 - R^2\right]dz\,d\theta$$

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$$= \frac{1}{2}\int_0^{2\pi}\int_R^{\sqrt{3}R}\left[4R^2-z^2-R^2\right]dz\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_R^{\sqrt{3}R}\left[3R^2-z^2\right]dz\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[3R^2 z - \frac{z^3}{3}\right]_R^{\sqrt{3}R}d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[3\sqrt{3}R^3 - \frac{3\sqrt{3}R^3}{3} - 3R^3 + \frac{R^3}{3}\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[2\sqrt{3}R^3 - \frac{9}{3}R^3 + \frac{1}{3}R^3\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[2\sqrt{3}R^3 + \frac{8}{3}R^3\right]d\theta$$

$$= \frac{1}{2}\left[2\sqrt{3}R^3\theta + \frac{8}{3}R^3\theta\right]_0^{2\pi} = \frac{1}{2}\left[4\pi\sqrt{3}R^3 + \frac{16\pi}{3}R^3 - 0\right]$$

$$= \boxed{2\pi\sqrt{3}R^3 + \frac{8\pi}{3}R^3}$$

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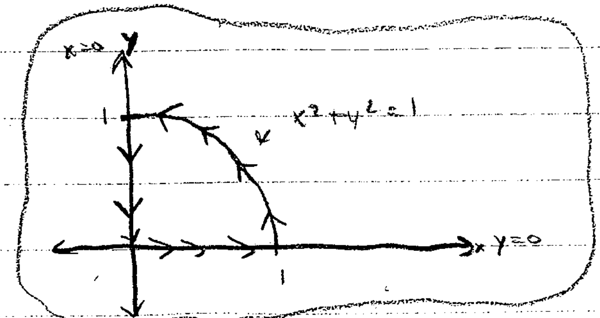
(4) $x^2+y^2=1$, $y=0$, $x=0$, $F = xy\hat{\imath} + xy\hat{\jmath}$

(a)



$$\boxed{r_1(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath}} \quad \text{for} \quad \boxed{0 \le t \le \pi/2}$$

from (0,1) to (0,0) at $t = \pi/2$, $r(t) = 1$

at $t = \pi$ $r(t) = 0$

$$\frac{0-1}{\pi - \pi/2} = \frac{-1}{\frac{\pi}{2}} = \frac{-2}{\pi}$$

$$\boxed{r_2(t) = \left(-\frac{2}{\pi}t + 2\right)\hat{\jmath}} \quad \text{for} \quad \boxed{\pi/2 \le t \le \pi}$$

from (0,0) to (1,0) at $t = \pi$, $r(t) = 0$

at $t = 2\pi$, $r(t) = 1$

$$\frac{1-0}{2\pi - \pi} = \frac{1}{\pi}$$

$$\boxed{r_3(t) = \left(\frac{1}{\pi}t - 1\right)\hat{\imath}} \quad \text{for} \quad \boxed{\pi \le t \le 2\pi}$$

(b)

$$\frac{dr_1(t)}{dt} = -\sin(t)\hat{\imath} + \cos(t)\hat{\jmath}$$

$$F_1(t) = (\cos(t)\sin(t))\hat{\imath} + (\cos(t)\sin(t))\hat{\jmath}$$

$$F \cdot \frac{dr_1(t)}{dt} = -\cos(t)\sin^2(t) + \cos^2(t)\sin(t)$$

$$\int_0^{\pi/2} (\cos^2(t)\sin(t) - \cos(t)\sin^2(t))\,dt$$

$$= \left[-\tfrac{1}{3}\cos^3(t) - \tfrac{1}{3}\sin^3(t)\right]_0^{\pi/2}$$

$$= \left[0 - \tfrac{1}{3} + \tfrac{1}{3} + 0\right] = 0$$

$F_2(t) = 0\hat{i} + 0\hat{j} = 0$

$\therefore \int_{\pi/2}^{\pi} F_2 \cdot dr_2 = 0$

$F_3(t) = 0\hat{i} + 0\hat{j} = 0$

$\therefore \int_{\pi}^{2\pi} F_3 \cdot dr_3 = 0$

$\therefore$ Flow $= 0 + 0 + 0 = \boxed{0}$

(c)

$F_1 \Rightarrow M = \cos(t)\sin(t)$

$N = \cos(t)\sin(t)$

$dy = \cos(t)$

$dx = -\sin(t)$

$\int_0^{\pi/2} M\,dy - N\,dx$

$= \int_0^{\pi/2} [\cos^2(t)\sin(t) + \cos(t)\sin^2(t)]\,dt$

$= [-1/3\cos^3(t) + 1/3\sin^3(t)]_0^{\pi/2}$

$= [0 + 1/3 + 1/3 + 0] = \underline{2/3}$

$F_2 \Rightarrow M = 0, N = 0 \Rightarrow F_2 = 0$

$\therefore \int_{\pi/2}^{\pi} [0 - 0]\,dt = \underline{0}$

$F_3 \Rightarrow M = 0, N = 0 \Rightarrow F_3 = 0$

$\therefore \int_{\pi}^{2\pi} [0 - 0]\,dt = \underline{0}$

$\therefore \boxed{\text{Flux} = 2/3}$