Problem 1.

(a) The surface $x+y+z=1$ describes a plane, + so we should expect the tangent plane to the surface be the plane itself. Observe if $g(x,y,z) = x+y+z$, then $g(x,y,z)=1$ defines surface so $\nabla g = \hat{i} + \hat{j} + \hat{k}$ is normal to surface + is the same @ every point.

Therefore tangent plane is $1(x-x_0) + 1(y-y_0) + 1(z-z_0) = 0$

$$\Leftrightarrow (x-0) + (y-1) + (z-0) = 0$$

$$\Leftrightarrow \boxed{x+y+z=1.}$$

(b) $df = (\nabla f \cdot \hat{u})\, ds$

where $\hat{u} = \frac{u}{|u|}$

$$\underline{u} = 2\hat{i} + 2\hat{j} + 2\hat{k} \Rightarrow |\underline{u}| = 2\sqrt{3} \quad \text{so} \quad \hat{u} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

$$\nabla f = e^x \cos(yz)\hat{i} - ze^x \sin(yz)\hat{j} - ye^x \sin(yz)\hat{k}$$

$$\Rightarrow \nabla f\big|_{(0,0,0)} = \hat{i}$$

$$\Rightarrow \nabla f \cdot \hat{u} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow df = \frac{1}{\sqrt{3}} ds$$

If $ds = 0.1$ then $\boxed{df = \frac{1}{10\sqrt{3}}}$

(c). $w(x,t) = \cos(2x + 2ct)$

$$\frac{\partial w}{\partial x} = -2\sin(2x+2ct) \qquad \frac{\partial w}{\partial t} = -2c\sin(2x+2ct)$$

$$\frac{\partial^2 w}{\partial x^2} = -4\cos(2x+2ct) \qquad \frac{\partial^2 w}{\partial t^2} = -4c^2\cos(2x+2ct)$$

so $$c^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2} \quad \checkmark$$

Problem 1 - con't

(d) ← Not on exam

Observe $\frac{x^2-2x+1}{x^2-2x-y+2} = \frac{(x-1)^2}{(x-1)^2-(y-1)}$

So to evaluate $\lim_{(x,y)\to(1,1)} \frac{x^2-2x+1}{x^2-2x-y+2}$

First, let $y=1$, then $\lim_{(x,y)\to(1,1)} \frac{x^2-2x+1}{x^2-2x-y+2} = 1$

Now, let $y=x$, then $\lim_{(x,y)\to(1,1)} \frac{x^2-2x+1}{x^2-2x-y+2} = \lim_{(x,x)\to(1,1)} \frac{(x-1)}{(x-1)-1}$

$= 0$

Since limits are different along paths $y=1$ + $y=x$, the limit must not exist.

Problem 2

$f(x,y) = x^2-3xy+5 \Rightarrow f(2,1) = 4-6+5 = 3$

(a) $L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$f_x(x,y) = 2x-3y \Rightarrow f_x(2,1) = 1$

$F_y(x,y) = -3x \Rightarrow f_x(2,1) = -6$

$\Rightarrow L(x,y) = 3 + (x-2) - 6(y-1)$

$= \boxed{x-6y+7}$

(b) Know $|E| \leq \frac{M}{2}(|x-2|+|y-1|)^2$

Where $M \geq (F_{xx}, F_{yy}, f_{yz})$

Problem 2 con't

Observe

$f_{xx}(x,y) = 2$

$f_{xy}(x,y) = -3$

$f_{yy}(x,y) = 0$ so $M \geq 3$ choose $M=3$ for lowest error bound.

$$\Rightarrow |E| \leq \frac{3}{2}\left(|x-2|+|y-1|\right)^2$$

$$\leq \frac{3}{2}\left(\frac{1}{10}+\frac{1}{10}\right)^2 = \frac{3}{2}\ \frac{4}{100} = \boxed{\frac{3}{50}}$$

(c)

Taylor expansion yields

$$T_2(x,y) = \underbrace{x-6y+7}_{L(x,y)} + \frac{1}{2}\left(2(x-2)^2 + 2(-3)(x-2)(y-1) + 0\right)$$

$$= \boxed{x^2 - 3xy + 5}$$

Problem 5

$T(x,y) = x^2 - y^2 - x$

(a) $T_x = 2x - 1$ critical pts when $\left.\begin{matrix} T_x = 0 \\ T_y = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} x = \frac{1}{2} \\ y = 0 \end{matrix}$ are critical pts

$T_y = -2y$

$T_{xx} = 2$

$T_{yy} = -2$

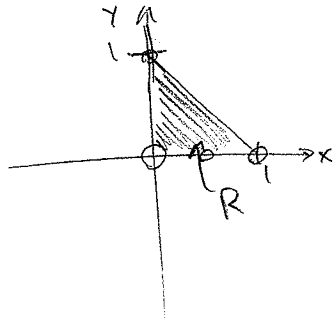
$T_{xy} = 0$

So $H = T_{xx}T_{yy} - T_{xy}^2 = -4 < 0$ so $\boxed{(\frac{1}{2},0) \text{ is saddle point.}}$

The function evaluated @ critical pt is $T(\frac{1}{2},0) = -\frac{1}{4}$

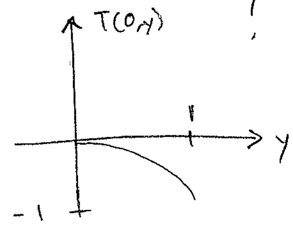
Problem 5 con't

(b) The region looks like



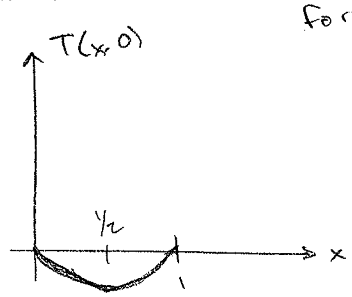
On $x=0$

$T(0,y) = -y^2$ so For $y \in [0,1]$ $T(0,y)$ is a max at $y=0$
a min at $y=1$



On $y=0$ $T(x,0) = x^2 - x = x(x-1)$ for $x \in [0,1]$

So



$T(x,0)$ is a min @ $x = \frac{1}{2}$

a max @ $x = 0, + 1$

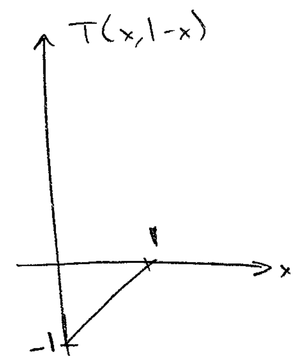
On $x+y=1$

write $y = 1-x \Rightarrow T(x,1-x) = x^2 - (1-x)^2 - x$

$= x^2 - (1 - 2x + x^2) - x$

$= x^2 - 1 + 2x - x^2 - x$

$= x - 1$



So for $x \in [0,1]$

$T(x,1-x)$ has its min at $x=0$

its max at $x=1$

Problem 5 cont

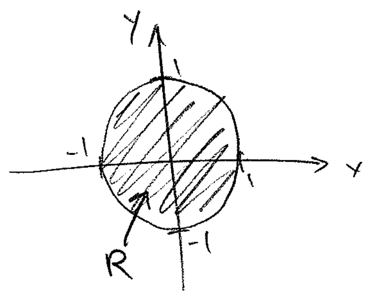
So, in summary, the boundary has the following local extrema

$(0,0)$	max value of	0
$(0,1)$	min value of	-1
$(\frac{1}{2},0)$	min value of	$-\frac{1}{4}$
$(1,0)$	max value of	0

So global max is 0 at points $(0,0)$ and $(1,0)$

global min is -1 at point $(0,1)$

(c) The region now looks like



In interior of R, we know $\exists$ a $\boxed{\text{saddle point @ } (\tfrac{1}{2},0)}$ from analysis in part (a), but this is not an extreme value. So we need to seek extrema of $T(x,y)$ on boundary. The boundary is given by $g(x,y) = x^2+y^2 = 1$

LM says $\nabla T = \lambda \nabla g$

$$\Rightarrow (2x-1)\hat{\imath} + (-2y)\hat{\jmath} = \lambda\left(2x\hat{\imath} + 2y\hat{\jmath}\right)$$

so 1.) $2x-1 = \lambda 2x$

$\Leftrightarrow 2x(1-\lambda) = 1$

2.) $-2y = \lambda 2y$ so if $y \neq 0$ $\lambda = -1$

then eq'n 1 says $x = \frac{1}{4}$

so the constraint says $y = \pm\sqrt{\frac{15}{16}} = \pm\frac{\sqrt{15}}{4}$

Problem 5 con't

if $y=0$, then constraint says $x = \pm 1$

$\therefore$ LM leads to the solution points

$$\boxed{\begin{array}{c} (1,0) \\ (-1,0) \\ \left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right) \\ \left(\frac{1}{4}, -\frac{\sqrt{15}}{4}\right) \\ \hline + \text{saddle @ } \left(\frac{1}{2}, 0\right) \end{array}}$$

So, the extreme values are

$$T(1,0) = 0$$

$$T(-1,0) = 2$$

$$T\left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right) = \left(\frac{1}{4}\right)^2 - \left(\frac{\sqrt{15}}{4}\right)^2 - \frac{1}{4} = \frac{1}{16} - \frac{15}{16} - \frac{1}{4}$$

$$= \frac{1-15-4}{16} = -\frac{18}{16}$$

$$T\left(\frac{1}{4}, -\frac{\sqrt{15}}{4}\right) = -\frac{18}{16}$$

Problem 3

(a) $V(r,h) = \pi r^2 h$

$$\Rightarrow \frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

So when $r = 10$ inches

$h = 8$ inches

we have $$\frac{dV}{dt} = 2\pi (10\text{in})(8\text{in}) \frac{dr}{dt} + \pi (10\text{in})^2 \frac{dh}{dt}$$

$$= \left(160\pi \text{ in}^2\right)\frac{dr}{dt} + \left(100\pi \text{ in}^2\right)\frac{dh}{dt}$$

So if $\frac{dr}{dt} = 3 \frac{\text{in}}{\text{min}}$ & $\frac{dh}{dt} = -5 \frac{\text{in}}{\text{min}}$

then $$\frac{dV}{dt} = 480\pi \frac{\text{in}^3}{\text{min}} - 500\pi \frac{\text{in}^3}{\text{min}} = \boxed{-20\pi \text{ in}^3/\text{min}}$$

Problem 4

(a) Let $F(x,y,z) = f(x,y) - z$. So surface defined by $F(x,y,z)=0$, Normal vector to surface is $\nabla F = f_x \hat{i} + f_y \hat{j} - 1\hat{k}$

$= -2x\hat{i} - 2y\hat{j} - 1\hat{k}$

$\Rightarrow$ Normal vec to surface @ point $(0,0,10)$ is $\boxed{\nabla F\big|_{(0,0,10)} = -\hat{k}}$

(b) $\boxed{\nabla R(x,y,z) = (2xy+4)\hat{i} + (x^2+2yz+4)\hat{j} + (y^2+1)\hat{k}}$

(c)

Using Normal vector to S as normal vector of tangent plane we have

$0(x-0) + 0(y-0) + (-1)(z-10) = 0 \Rightarrow \boxed{z = 10.}$

Can also see from drawing paraboloid

(d) We want to minimize $D_{\hat{u}}R$ for some $\hat{u}$ in the tangent plane to the surface @ $(0,0,10)$. We can write any one of the $\hat{u}$ as $\hat{u} = \cos\theta\hat{i} + \sin\theta\hat{j}$ for some $\theta \in [0, 2\pi)$ since the tangent plane is $\parallel$ to the xy-plane.

$\therefore D_{\hat{u}}R\big|_{(0,0,10)} = (\hat{u})\cdot(\nabla R)\big|_{(0,0,10)}$

$= (\cos\theta\hat{i} + \sin\theta\hat{j})\cdot(4\hat{i} + 4\hat{j} + \hat{k})$

$= 4\cos\theta + 4\sin\theta$

Let $g(\theta) = D_{\hat{u}}R$

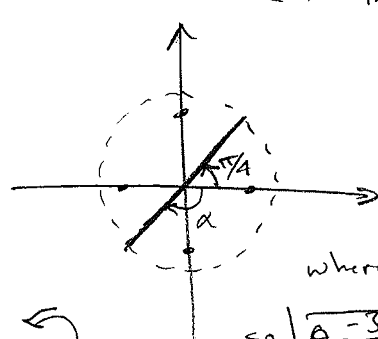
Then $g'(\theta) = 0$ gives critical pnt on $g(\theta)$. This occurs if $-4\sin\theta + 4\cos\theta = 0$

$\Longleftrightarrow \tan\theta = 1$

So $\theta = \tan^{-1}(1)$

or $\sin\theta = \cos\theta$

Observe from unit circle that this occurs @ two angles...



(scalar) so $g(\theta)\big|_{\pi/4} = 4\sqrt{2}$ while $g(\theta)\big|_{-3\pi/4} = -4\sqrt{2}$

where $\alpha = (2\pi - \frac{5\pi}{4}) = -\frac{3\pi}{4}$

2^{nd} deriv. test is inconclusive so need to eval $g(\theta)$...

so $\boxed{\theta = -\frac{3\pi}{4} \text{ gives minimizing } \hat{u}}$