

- Basics
 - An ODE is an equation involving y , t , and the various derivatives $\frac{d^k y}{dt^k}$ up to some order n .
 - y is the dependent variable, t is the independent variable, n is the order of the ODE.
 - A solution is any function $y(t)$ that “works” when plugged into the ODE.
 - An IVP is an ODE along with an initial condition (that specifies the value of y at some particular value of t).
- Solutions
 - For first order equations, as long as the rate function is nice ($f(t, y)$ is continuous in some region) then solutions exist.
 - Solutions to an ODE are not unique until an initial condition is specified.
 - If $\frac{\partial f}{\partial y}$ is continuous in a region around a given initial condition, then the solution to the IVP is unique. The existence and uniqueness of this solution is guaranteed only locally, however.
 - The general solution to an ODE is one that can satisfy any given initial condition.
 - A particular solution is one that satisfies only one initial condition (*i.e.* it is one out of the infinitely many solutions contained in the general solution).
- Three-pronged attack
 - Analytic: try to solve the equation directly. Gives exact function as answer, but may not be possible for many equations.
 - Qualitative: try to determine general properties of solutions, without actually solving. Works on pretty much any equation and doesn't require hard math, but doesn't give any exact values, or even approximate values (just general properties).
 - Numerical: try to get approximate values of the solution(s). Gives actual values and works on most equations, but gives solution values only at certain values of t for particular solutions (*i.e.* requires an initial condition). Gives no conceptual understanding. Requires a computer, in reality.
- Analytic
 - An equation is separable if it can be written as $y' = g(t)h(y)$, for some functions g, h . Can be solved by separation of variables:
 - * rewrite as $\frac{1}{h(y)} \frac{dy}{dt} = g(t)$
 - * integrate with respect to t : $\int \frac{1}{h(y)} \frac{dy}{dt} dt = \int g(t) dt$
 - * which is equivalent to: $\int \frac{1}{h(y)} dy = \int g(t) dt$
 - * evaluate the integrals and, if possible, solve for $y(t)$.
 - * make sure your “general” solution actually includes equilibrium solutions that might have been “lost” by dividing by $h(y)$.

- An equation is linear if it can be written as $y' + p(t)y = q(t)$ (or, more generally, $a_n(t)y^{(n)} + \dots + a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$). A linear equation is homogeneous if there is no function of t on its own — *i.e.* $q(t) \equiv 0$ (or $f(t) \equiv 0$). Solutions to linear equations have special properties.
 - * Given any number of solutions, y_1, \dots, y_k , to the homogeneous part of the equation, the function $y(t) = c_1y_1 + \dots + c_ky_k$ is also a solution (to the homogeneous part) for any set of constants $\{c_i\}$.
 - * Given a solution y_h to the homogeneous part of the equation, and a solution y_p to the full equation, the function $y(t) = y_h + y_p$ is also a solution to the full equation. Moreover, if y_h is the general homogeneous solution, then $y = y_h + y_p$ is the general solution (to the full problem).

Therefore one way to solve first order linear equations is the Euler-Lagrange (Variation of Parameters) approach:

- * Find the general solution y_h of the homogeneous part by separation of variables (or by inspection)
- * Assume a form of the particular solution y_p in which the constant C in y_h is allowed to be a function $C(t)$
- * Substitute this choice of y_p into the equation to get an equation for $C'(t)$
- * Solve this equation to get $C(t)$ and hence y_p
- * By linearity, the general solution is $y = y_h + y_p$

However, another approach is to use Integrating Factors:

- * For the equation $y' + p(t)y = q(t)$, define the integrating factor $\mu = e^{\int p(t)dt}$
 - * Multiply the equation by $\mu(t)$
 - * The left-hand side can now be rewritten as $\frac{d}{dt}(\mu(t)y)$ (check back to make sure!)
 - * Integrate both sides with respect to t (don't forget the constant!)
 - * Solve for $y(t)$
- Qualitative: evaluating $f(t, y)$ at various points in the (t, y) -plane gives the slopes of the solutions of $y' = f(t, y)$ that happen to pass through those points. Drawing small lines with the appropriate slopes at a number of points creates a slope field, which gives a visual idea of how the solutions behave.
 - Numerical: you have seen the simplest numerical method known as Euler's Method for obtaining approximate solutions to an IVP at some given time T :
 - Choose a stepsize h . At the 0^{th} step (the start), you have t_0 and y_0 given by the initial condition. For $n = 1, 2, \dots, N$ (where $N = T/h$ — the smaller the stepsize, the more steps you have to take) repeat the following procedure:
 - $t_n = t_{n-1} + h$
 - $y_n = y_{n-1} + hf(t_{n-1}, y_{n-1})$
 - In words, the time is advanced in equal steps; the solution is advanced by evaluating the slope where you currently are and stepping forward from the current value a distance h with that slope.
 - Euler's Method is $\mathcal{O}(h)$ or "first order" ($h = h^1$), meaning that there error in the approximation is proportional to h .

In general, numerical methods are classified as p^{th} order ($\mathcal{O}(h^p)$) if their error is proportional to h^p . Generally, higher-order is better.