SUMMARY OF SOLUTION METHODS FOR FIRST ORDER ODES

ODEs with the normal form $y' = f(t, y)$ can be solved analytically only in special cases. Below is a summary of those cases that we have covered in class:

$y' = f(t)$ — Direct integration (with respect to t).

$y' = f(t) \cdot h(y)$ — ***Separable equation***; write as $\int \frac{1}{h(y)} dy = \int f(t)\, dt$ and integrate.

$y' = h\left(\frac{y}{t}\right)$ — ***Euler - homogeneous equation***. Test: change $\begin{cases} y \to ky \\ t \to kt \end{cases}$ in the RHS, and see if it made any difference. Set $u = \frac{y}{t} \Rightarrow y = t\,u \Rightarrow y' = u + t\,u'$; gives separable ODE for u.

$y' = f(t) \cdot y + g(t)$ — ***Linear, inhomogeneous equation***; two main solution methods:

Variation of parameter:
Solve $Y' = f(t) \cdot Y$ (separable). Then change constant C in that solution to $c(t)$. Plug into original ODE; will give an ODE for $c(t)$ that can be solved by direct integration.

Integrating Factor:
Write ODE as $y' + p(t) \cdot y = q(t)$. Find a $P(t)$ such that $P'(t) = p(t)$, multiply by integrating factor $e^{P(t)}$. Gives

$$\underbrace{e^{P(t)}\, y' + e^{P(t)} p(t)\, y}_{\frac{d}{dt}\left(e^{P(t)} \cdot y\right)} = e^{P(t)}\, q(t)$$

Now solve for y.

$y' = f(t) \cdot y + g(t) \cdot y^{\alpha}$ — ***Bernoulli equation***. If $\alpha = 0$ or $\alpha = 1$, solve as before (linear or separable, resp.). Else, set $u = y^{1-\alpha}$, i.e. $y = u^{1/(1-\alpha)}, y' = \ldots$ Gives a linear first order ODE for u.

$y' = f(t) + g(t) \cdot y + h(t) \cdot y^2$ — ***Riccati equation***. If we somehow know one solution $y_1(t)$, then we can find the general solution by setting $y(t) = y_1(t) + 1/v(t)$; will give a linear first order ODE for $v(t)$.

Some additional cases that can also be solved by simple substitutions:

$y' = f(at + by + c)$ — Set $u = at + by + c$, get a separable ODE for u.

$y' = \dfrac{a_1 y + b_1 t}{a_2 y + b_2 t}$ — Special case of Euler-homogeneous equation; see above.

$y' = \dfrac{a_1 y + b_1 t + c_1}{a_2 y + b_2 t + c_2}$ — If lines not parallel, find intersection (t_0, x_0), change variables $\hat{t} = t - t_0$, $\hat{y} = y - y_0$. Gives Euler-homogeneous equation for $\hat{y}(\hat{t})$.