ROOTS OF POLYNOMIALS

Assume that we have normalized the polynomial so that the leading coefficient is equal to one:

$$p_n(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n \qquad (1)$$

Roots; explicit formulas:

Allowing for multiple roots and for complex roots, $p_n(x)$ has precisely n roots (solutions to $p_n(x) = 0$); we denote them by $x_1, x_2, \ldots, x_n$.

$\underline{n = 1}$: $p_1(x) = x + a_1$, root $x_1 = - a_1$.

$\underline{n = 2}$: $p_2(x) = x^2 + a_1 x + a_2$, roots $x_{1,2} = -\frac{a_1}{2} \pm \sqrt{(\frac{a_1}{2})^2 - a_2}$.

There are also formulas for $n = 3$ and $n = 4$, but they are too complicated to be useful to us.

Factors:

With the roots $x_1, x_2, \ldots, x_n$, the polynomial can be written (factored)

$$p_n(x) = (x - x_1)(x - x_2) \cdot \ldots \cdot (x - x_n) \quad . \qquad (2)$$

Relations between roots and coefficients:

Multiplying out the product in (2) gives

$$p_n(x) = x^n - (x_1 + x_2 + \ldots + x_n) x^{n-1} + \ldots + (-1)^n x_1 x_2 \cdot \ldots \cdot x_n$$

Comparing with (1) gives the relations

$$a_1 = -(x_1 + x_2 + \ldots + x_n) = -\sum_{i=1}^{n} x_i \ ,$$
$$a_n = (-1)^n x_1 x_2 \cdot \ldots \cdot x_n = (-1)^n \prod_{i=1}^{n} x_i$$

Guessing roots of high order polynomials with integer coefficients:

If all the coefficients $\{a_i\}$ of the polynomial in (1) are integers, then the only possible integer roots must be of the form $x_i = \pm\{\text{an integer factor in } a_n\}$.

Example: Given $p_n(x) = x^n + \ldots - 5$, the only feasible guesses for an integer root are ±1 and ±5. (Guessing on any other integer values - or even rational values - would be a complete waste of effort.)

This result is a special case of the following: If the leading coefficient is not 1 , but the polynomial is of the form $p_n(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ where all the coefficients are integers, then the only possible *rational* roots are of the form $\pm\frac{q}{r}$ where q is a factor in a_n and r is a factor in a_0.

Horner's scheme:

This algorithm performs very effectively several useful tasks

- Evaluate $p_n(x)$ for any specific value of x,
- Test if a guess $x = x_g$ is a root to $p_n(x) = 0$,
- If we have found a root x_1 to $p_n(x)$, obtain the coefficients of the polynomial $p_{n-1}(x)$ which satisfies $p_n(x) = (x - x_1)\, p_{n-1}(x)$ ('Divide out' a root).

Applied to $p_n(x)$, as given by (1), and with $x = x_g$, the algorithm goes as follows:

$$\begin{array}{ccccccccccc|c}
1 & & a_1 & & a_2 & & a_3 & & \ldots & & a_n & \underline{x_g} \\
 & & 1\cdot x_g & & (a_1+x_g)\cdot x_g & & \ldots & & \ldots & & \ldots & \\
\hline
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & & \nearrow & \downarrow & \\
1 & & a_1+x_g & & \ldots & & \ldots & & \ldots & & \boxed{p_n(x_g)} &
\end{array}$$

The steps are as follows:

- Enter in the top row the coefficients of $p_n(x)$ followed by the x-value x_g
- Start from the left. Add the entries in the first and second row (for the first column, the second row counts as zero). Put the result under the line. Multiply the result by x_g and let that be the next entry in the second row. Repeat this until we have come to the last (boxed) position.

We can now read off the value of $p_n(x_g)$. The remaining entries of the bottom row - from left to right - are coefficients in the polynomial $p_{n-1}(x)$, satisfying

$$p_n(x) = (x - x_g)\, p_{n-1}(x) + p_n(x_g).$$

This last result is usually of most interest when we have found a value x_g such that $p_n(x_g) = 0$ (i.e. x_g is a root to $p_n(x) = 0$). Then the bottom row, up to but not including its last entry, contains the coefficients of the polynomial that is obtained when the found root is divided away.

<u>Example:</u> Find the roots of $p_3(x) = x^3 - 19\,x + 30 = 0$.

<u>Solution:</u> The only integer values worth trying as roots are ±1, ±2, ±3, ±5, ±6, ±10, ±15 and ±30. So we do this with Horner's scheme (note that we need to include a coefficient 0 for the missing x^2-term):

Try $x = 1$:

$$\begin{array}{rrrr|l} 1 & 0 & -19 & 30 & \underline{\;1\;} \\ & 1 & 1 & -18 & \\ \hline 1 & 1 & -18 & 12 & \end{array}$$

Didn't work;

Try $x = -1$:

$$\begin{array}{rrrr|l} 1 & 0 & -19 & 30 & \underline{\;-1\;} \\ & -1 & 1 & 18 & \\ \hline 1 & -1 & -18 & 48 & \end{array}$$

Still no luck;

Try $x = 2$:

$$\begin{array}{rrrr|l} 1 & 0 & -19 & 30 & \underline{\;2\;} \\ & 2 & 4 & -30 & \\ \hline 1 & 2 & -15 & 0 & \end{array}$$

YES!! $x_1 = 2$.

So $p_3(x) = (x - 2)\,(x^2 + 2\,x - 15)$

We now find the remaining roots by the formula for the quadratic:

$$x_{2,3} = -1 \pm \sqrt{1+15} \;=\; -1 \pm 4\ .$$

In conclusion: the roots are $x_1 = 2$, $x_2 = 3$, $x_3 = -5$ and the polynomial can be factored $p_3(x) = (x - 2)\,(x - 3)\,(x + 5)$.