Vector Space Practice

For the following examples, determine if each set is a vector space. If so, find a basis and thus determine the space's dimension.

1. Let P be the set of all quadratic polynomials of the form $p(x) = x^2 + a_1 x + a_0$. The operations $+$ and $\cdot$ are defined by

$$p(x) + q(x) = (x^2 + a_1 x + a_0) + (x^2 + b_1 x + b_0) = x^2 + (a_1 + b_1)x + a_0 b_0$$
$$kp(x) = k \cdot (x^2 + a_1 x + a_0) = x^2 + k a_1 x + k^2 a_0$$

2. Let T_n be the set of all $n \times n$ matrices M such that the "trace" of M — the sum of the diagonal elements of M — is zero. Mathematically, $T_n = \{M_{n\times n} : \text{tr}(M) = \sum_j M_{j,j} = 0\}$. The operations $+$ and $\cdot$ are defined in the standard way for matrices.

3. Let T_n be the set of all $n \times n$ matrices M such that the "trace" of M — the sum of the diagonal elements of M — is one. Mathematically, $T_n = \{M_{n\times n} : \text{tr}(M) = \sum_j M_{j,j} = 1\}$. The operations $+$ and $\cdot$ are defined in the standard way for matrices.

4. Let $\mathcal{Q}$ be the set of all 2×2 arrays of the symbols $\clubsuit, \diamondsuit, \heartsuit, \spadesuit$. The operation $+$ is defined in the standard (matrix) element-by-element manner, according to the rule:

$+$	$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\spadesuit$
$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\spadesuit$	$\clubsuit$
$\diamondsuit$	$\heartsuit$	$\spadesuit$	$\clubsuit$	$\diamondsuit$
$\heartsuit$	$\spadesuit$	$\clubsuit$	$\diamondsuit$	$\heartsuit$
$\spadesuit$	$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\spadesuit$

So, for example,

$$\left[\begin{array}{c|c} \heartsuit & \clubsuit \\ \hline \heartsuit & \spadesuit \end{array}\right] + \left[\begin{array}{c|c} \spadesuit & \clubsuit \\ \hline \diamondsuit & \heartsuit \end{array}\right] = \left[\begin{array}{c|c} \heartsuit & \diamondsuit \\ \hline \clubsuit & \heartsuit \end{array}\right]$$

The operation $\cdot$ is defined for integer scalars according to the following rule:

The elements of the array are numbered as

$$\left[\begin{array}{c|c} 0 & 1 \\ \hline 3 & 2 \end{array}\right]$$

Determine the value of the number of the element multiplied by $|k|$, modulo 4:

$$\left[\begin{array}{c|c} 0 & |k| \bmod 4 \\ \hline 3|k| \bmod 4 & 2|k| \bmod 4 \end{array}\right]$$

The symbol in each position is the symbol that was in the position given by that number. For example, for $k = 2$,

$$\left[\begin{array}{c|c} 0 & |k| \bmod 4 \\ \hline 3|k| \bmod 4 & 2|k| \bmod 4 \end{array}\right] = \left[\begin{array}{c|c} 0 & 2 \bmod 4 \\ \hline 6 \bmod 4 & 4 \bmod 4 \end{array}\right] = \left[\begin{array}{c|c} 0 & 2 \\ \hline 2 & 0 \end{array}\right]$$

and so

$$2 \cdot \left[\begin{array}{c|c} \heartsuit & \clubsuit \\ \hline \heartsuit & \spadesuit \end{array}\right] = \left[\begin{array}{c|c} \heartsuit & \spadesuit \\ \hline \spadesuit & \heartsuit \end{array}\right]$$

because $\heartsuit$ was in the 0 position and $\spadesuit$ was in the 2 position.

Similarly, for $k = -7$,

$$\left[\begin{array}{c|c} 0 & 7 \bmod 4 \\ \hline 21 \bmod 4 & 14 \bmod 4 \end{array}\right] = \left[\begin{array}{c|c} 0 & 3 \\ \hline 1 & 2 \end{array}\right]$$

and so

$$-7 \cdot \left[\begin{array}{c|c} \heartsuit & \clubsuit \\ \hline \heartsuit & \spadesuit \end{array}\right] = \left[\begin{array}{c|c} \heartsuit & \heartsuit \\ \hline \clubsuit & \spadesuit \end{array}\right]$$

because $\heartsuit$ was in the 1 position and $\clubsuit$ was in the 3 position.

5. Let S be the set of all solutions of the linear ODE $\mathcal{L}[y(x)] = f(x)$, such that $y(0) = 0$ and $y(\pi) = 0$.