ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructors' name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and box in your final answer. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, calculators and ANY electronic devices are NOT permitted. A $8'' \times 11''$, two-sided, sheet of notes is allowed.

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1. (20 points)
   (a) (8 points) Solve the IVP
   $$y' = \cos(t)e^{\sin(t)-y}, \qquad y(0) = 1.$$
   (b) (12 points) Find the general solution of
   $$y' + \frac{1}{t}y = \frac{1}{t^2+1}.$$
2. (20 points) Consider the differential equation
   $$y'' - 2y' + y = \frac{e^t}{t}$$
   (a) (8 points) Find the general solution of the homogeneous equation.
   (b) (10 points) Find a particular solution of the nonhomogeneous equation.
   (c) (2 points) Find the general solution of the nonhomogeneous equation.
3. (25 points) Consider the system of differential equations
   $$\begin{cases} x' = x + y \\ y' = -x + y \end{cases}$$
   (a) (20 points) Find the general solution. Make sure your solution is real.
   (b) (5 points) Sketch the phase portrait for the system.
4. (40 points) Consider the autonomous system of DEs
   $$\begin{cases} x' = y \\ y' = 1 - x^2 - y \end{cases}$$
   (a) (10 points) Determine all the equilibrium points and the Jacobian matrix corresponding to each equilibrium point.
   (b) (15 points) Find the eigenvalues of the Jacobian and use them to determine the stability and the type of equilibrium (for example, saddle, spiral, center, star, etc) for each fixed point of the nonlinear system.
   (c) (15 points) Sketch the phase portrait of the nonlinear system including all the nullclines and the directions ON and BETWEEN nullclines. Make sure your figure is big and clear.
5. (26 points) Consider the system of equations
   $$\begin{aligned} x + y + z &= 2, \\ y + z &= 1, \\ -x + 2y + z &= 0. \end{aligned} \tag{1}$$
   (a) (9 points) Write the system in the form $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and find the determinant of the matrix $\mathbf{A}$.
   (b) (10 points) Determine if Equations (1) have a solution and, if they do, find all the solutions.

PLEASE TURN OVER. QUESTION 5 CONTINUES ON NEXT PAGE.

(c) (7 points) Determine whether or not the vectors

$$\vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \qquad \vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{\mathbf{u}}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

are linearly independent.

6. (20 points) Suppose that

$$\ddot{x} - 2\dot{x} + 5x = f(t).$$

(a) (5 points) Find the homogeneous solution, $x_h(t)$.

(b) (10 points) Using the method of undetermined coefficients, find a particular solution $x_p(t)$ when $f(t) = 25t^2$.

(c) (5 points) Using the method of undetermined coefficients, write out the **form** of the particular solution $x_p(t)$ when $f(t) = t\cos(2t)$. Do **not solve** for any constants, just write down the form of $x_p(t)$ (for example "$x_p(t) = \alpha t + \beta$, where α and β are constants.")

7. (25 points) A cell culture consists of two types of cells, healthy cells and sick cells. Healthy cells reproduce at a rate proportional to the number of healthy cells with a constant of proportionality of 0.1, and sick cells die at a rate proportional to the number of sick cells, with a constant of proportionality 0.2 (with time in units of days).

(a) (5 points) Write differential equations for the number of healthy cells $h(t)$ and sick cells $s(t)$ and find the general solutions.

(b) (10 points) At time $t = 0$, there are twice as many sick cells as there are healthy cells. Find the time T at which there is an equal number of healthy and sick cells.

(c) (10 points) Suppose that the rate at which the healthy cells reproduce now depends on the amount of sunlight $L(t)$, which is modeled as $L(t) = 1 + \sin(2\pi t)$. The proportionality factor, which before was 0.1, is now $0.1L(t)$. Find $h(t)$ and describe what happens with the number of healthy cells $h(t)$ as $t \to \infty$.

8. (24 points) True/False questions. No explanations are needed. Make sure to write down whole words TRUE or FALSE not just T and F. Each question is worth 3 points.

(a) The vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}$ are linearly independent.

(b) The set of solutions to $y'' - 2ty' + y = 0$ is a vector space (usual addition and scalar multiplication are assumed).

(c) The linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ always has a solution.

(d) If 2 is an eigenvalue of an invertible matrix $\mathbf{A}$, then $\frac{1}{2}$ must be an eigenvalue of $\mathbf{A}^{-1}$.

(e) There exist 4 linearly independent vectors in $\mathbb{R}^3$.

(f) For linear system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A}$ is a 2 by 2 matrix, if $|A| = -2$, then equilibrium $(0, 0)$ is always a saddle.

(g) If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are invertible square matrices, $((\mathbf{AB})^T\mathbf{C})^{-1} = \mathbf{C}^{-1}(\mathbf{A}^{-1})^T(\mathbf{B}^{-1})^T$.

(h) If $|\mathbf{AB}| = 3$ where $\mathbf{A}$ and $\mathbf{B}$ are both square matrices, then $\mathbf{A}$ must be invertible.

IF YOU ARE FINISHED AND HAVE TIME, CHECK YOUR ANSWERS.