1) a) IVP $y' = \cos(t)\, e^{\sin(t) - y}$, $y(0) = 1$ ①

SOV: $\dfrac{dy}{dt} = \cos(t)\, e^{\sin(t)}\, e^{-y}$

$$\int dy\, e^{y} = \int \cos(t)\, e^{\sin(t)}\, dt$$

$$e^{y} = e^{\sin(t)} + c$$

$$y = \ln\left(e^{\sin(t)} + c\right)$$

Find c

$$e^{1} = e^{\sin(0)} + c \implies c = e - 1$$

$$\therefore\ y = \ln\left(e^{\sin(t)} + e - 1\right)$$

b) General solution: $y' + \frac{1}{t} y = \frac{1}{t^2+1}$

IF: $p(t) = \frac{1}{t} \implies P(t) = \ln t \implies e^{P(t)} = e^{\ln t} = t$

DE becomes

$$\int (ty)' = \int \frac{t}{t^2+1}\, dt = \frac{1}{2} \int \frac{dt^2}{t^2+1} = \frac{1}{2} \ln(t^2+1) + C_1$$

$$\implies ty = \frac{\ln(t^2+1) + C_2}{2}$$

$$\therefore\ y = \frac{\ln(t^2+1) + c}{2t}$$

2) $A = \begin{pmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

a) i) RREF

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 1 & 1 & -1 & 1 & 3 \end{array}\right] \xrightarrow{R_3 = R_3 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2 = R_2 + R_3]{R_1 = R_1 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

ii) Pivots? = 3 Rank? = 3 # solutions? $= \infty$ (consistent)

iii) General soln. to $A\underline{x} = \underline{b}$: x_3 is free, $x_3 = s$

$$\begin{aligned} x_1 - 3s &= 0 \\ x_2 + 2s &= 2 \\ x_3 &= s \\ x_4 &= 1 \end{aligned} \quad \leadsto \quad \begin{aligned} x_1 &= 3s \\ x_2 &= 2 - 2s \\ x_3 &= s \\ x_4 &= 1 \end{aligned} \quad \leadsto \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

b) General soln to $A\underline{x} = 0$? RREF:

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 & \cancel{2}0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{aligned} x_1 &= 3s \\ x_2 &= -2s \\ x_3 &= s \\ x_4 &= 0 \end{aligned} \quad \leadsto \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

~~inconsistent! No solutions.~~

3) $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad A^{-1} = \begin{pmatrix} 1 & 1 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

a) $A\underline{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \underline{x} = A^{-1}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

b) $B\underline{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \longrightarrow \left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$ Inconsistent! No solutions.

c) $ZA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} \Rightarrow Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 1 & -3 \\ 0 & -2 & 2 \end{pmatrix}$

d) $WA + BWA = X, \quad X = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$(I+B)WA = X$

$W = (I+B)^{-1} X A^{-1}$, if $I+B$ invertible.

$I + B = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = A \Rightarrow (I+B)^{-1} = A^{-1}$

$\therefore W = A^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A^{-1} = A^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 \\ 1 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

4) $\underline{V}$ is set of all infinite sequences $(S_n)_{n=1}^{\infty} = (S_1, S_2, \dots)$

$\underline{W}$ is set of all sequences that satisfy $S_{\ell+1} = S_\ell + S_{\ell-1}$.

a) Show $\underline{W}$ is subspace of $\underline{V}$.

- 0 element? $\vec{O} = (0, 0, \dots)$, satisfies $\vec{O}_{\ell+1} = \vec{O}_\ell + \vec{O}_{\ell-1}$, so $\vec{O} \in \underline{W}$
- Closed under multiplication?

  If $\vec{v} \in \underline{W}$, is $\lambda \vec{v}$?

  $(\lambda v)_{\ell+1} = \lambda v_{\ell+1} = \lambda v_\ell + \lambda v_{\ell-1} = (\lambda v)_\ell + (\lambda v)_{\ell-1}$, so $\lambda \vec{v} \in \underline{W}$

  ✓
- Addition? $\vec{u}, \vec{v} \in \underline{W}$

  $(u+v)_{\ell+1} = u_{\ell+1} + v_{\ell+1} = u_\ell + u_{\ell-1} + v_\ell + v_{\ell-1} = (u+v)_\ell + (u+v)_{\ell-1}$

  so $\vec{u} + \vec{v} \in \underline{W}$

  ✓

b) Show 2 dimensional.

$e^{(1)} = (1 \; 0 \; 1 \; 1 \; 2 \; 2 \; \dots)$

$e^{(2)} = (0 \; 1 \; 1 \; 2 \; 2 \; \dots \;)$

The vectors form a basis:

$\vec{u} = u_1 e^{(1)} + u_2 e^{(2)}$

(we only need to specify the first 2 elements of $\underline{u}$)

5) PE_4 is set of all polynomials of form $v = a_0 + a_2 t^2 + a_4 t^4$

Equivalent to space of 3-D vectors, $\vec{v} = \begin{pmatrix} a_0 \\ a_2 \\ a_4 \end{pmatrix}$

a) Dimension of PE_4? 3

b) Write as vectors

$$V_1 = 1 + t^2 + t^4 = (1 \quad 1 \quad 1)^T$$
$$V_2 = -1 + t^4 = (-1 \quad 0 \quad 1)^T$$
$$V_3 = -2 + t^2 + 4t^4 = (-2 \quad 1 \quad 4)^T$$

c) Basis for PE_4? Check linear independence:

expand along 2nd row $\longrightarrow$

$$\begin{vmatrix} 1 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{vmatrix} = (-1)\begin{vmatrix} -1 & -2 \\ 1 & 4 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = \begin{matrix} (-1)(-4+2) \\ +(-1)(1+1) \end{matrix} = 0$$

~~$\det = -3 \neq 0 \Rightarrow$ lin indep $\Rightarrow$ basis~~ Not lin. indep, Not basis

d) Is $v = t^4$ in Span$\{v_1, v_2, v_3\}$? ~~Yes since $v \in PE_4$ & $\{v_1, v_2, v_3\}$ are basis for PE_4.~~

Need $c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Inconsistent $\Rightarrow$ no solutions!

$\therefore t^4$ is not in Span$\{v_1, v_2, v_3\}$.

6) T/F:

a) A is 4×3. $A\underline{x} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ always has soln. True ($\underline{x}=0$)

b) A is $n\times n$ w/ lin. indep. cols. RREF of A is identity mtx.
True

c) $\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}3&4\\4&5\\5&6\end{pmatrix} = \begin{pmatrix}3&4\\4&5\\5&6\end{pmatrix}$ False

$\uparrow$ 2×2 $=I$, but 3×2 dimensions don't match

d) A, B, C invertible & square. $((AB)^TC)^{-1} \overset{?}{=} C^{-1}(A^{-1})^T(B^{-1})^T$

$((AB)^TC)^{-1} = C^{-1}((AB)^T)^{-1} = C^{-1}(B^TA^T)^{-1} = C^{-1}(A^T)^{-1}(B^T)^{-1}$

$= C^{-1}(A^{-1})^T(B^{-1})^T$ ✓

True

e) $y' = 4y^{\cancel{3/2}\,2/3}$ $y(0)=0$ has unique soln.

~~Picards thm: $f(y)=4y^{3/2}$ ← Not continuous at $y=0$~~

False (Better: $y(t)=0$, $y=\frac{64}{27}t^3$)

f) $y' = e^{y-t^2-1}$ is separable. $\cancel{e^{-y}\,dy = e^{-t^2-1}\,dt}$

$e^{-y}y' = e^{-t^2-1}$ True

g) $y''+e^ty=\sqrt{t}$ is linear DE. True

h) $y(t)=e^{t^2}$ solves $y''-2ty'-2y=0$

$y'=2te^{t^2}$ $y''=2e^{t^2}+4t^2e^{t^2} \Rightarrow y''-2ty'-2y=0$

True

6) i) General soln to $x'' + 2x' + 10x = 5\cos(3t)$ becomes unbounded as $t \to \infty$.

$r^2 + 2r + 10 = 0 \Rightarrow r = -1 \pm 3i$

$b = 2 > 0 \Rightarrow$ False (no resonance) (solns are $\{e^{-t}\cos(3t), e^{-t}\sin(3t)\}$)

j) If $|AB| = 3$, A, B square matrices. A must be invertible.

$|AB| = |A||B| = 3 \Rightarrow |A| \neq 0 \Rightarrow A$ invertible.

True

7) $ty'' - (t+2)y' + 2y = t^3$

a) · Order: 2

· Linear?: yes

· Autonomous? no

· Homogeneous? no

e) · Constant coefficient? no

b) Homogeneous DE: $ty'' - (t+2)y' + 2y = 0$

Show $y_1 = e^t$ is soln: $y_1' = e^t$, $y_1'' = e^t$

$$te^t - (t+2)e^t + 2e^t = te^t - te^t - 2e^t + 2e^t = 0 \checkmark$$

c) Let $y_2 = z(t)y_1(t)$. Then if y_2 is solution:

i) $y_2' = z'y_1 + zy_1' = z'e^t + ze^t$

$y_2'' = z''e^t + z'e^t + z'e^t + ze^t$

substitute

$$\Rightarrow tz''e^t + tz'e^t + tz'e^t + tze^t - tz'e^t - tze^t - 2z'e^t - 2ze^t + 2ze^t = 0$$

$$tz''e^t + tz'e^t - 2z'e^t = 0$$

$$tz'' + (t-2)z' = 0 \checkmark$$

ii) Let $u = z' \Rightarrow tu' + (t-2)u = 0$. Solve w/ SOV. to get

$u = t^2e^{-t}$, or $z' = t^2e^{-t}$

IBP to get

$z = -t^2e^{-t} - 2te^{-t} - 2e^{-t}$

iii) $y_2 = zy_1 = -t^2 - 2t - 2$

d) Show y_1, y_2 span soln space. WKT y_1 and y_2 are solutions.

Need to show linear independence:

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & -t^2-2t-2 \\ e^t & -2t-2 \end{vmatrix} = -2te^t - 2e^t + t^2e^t + 2te^t + 2e^t$$

$= t^2e^t \neq 0$ unless $t = 0$, so linearly independent.

7) e) General homogeneous solution:

$$y_h = c_1 y_1 + c_2 y_2 = c_1 e^t + c_2(t^2+2t+2) \quad \text{Basis} = \{e^t, t^2+2t+2\}$$

f) Particular soln to nonhomo ~~eqn~~ eqn? Use VOP:

$$y_p = v_1(t)\, y_1(t) + v_2(t)\, y_2(t)$$

Need DE in std. form $y'' + p(t)y' + q(t)y = f(t)$:

$$y'' + \left(-1 - \frac{2}{t}\right)y' + \frac{2}{t}y = t^2 \Rightarrow p = -1 - \frac{2}{t}, \quad q = \frac{2}{t}, \quad f = t^2$$

Setup system

$$y_1 v_1' + y_2 v_2' = 0$$

$$y_1' v_1' + y_2' v_2' = f$$

$$e^t v_1' - (t^2+2t+2)v_2' = 0 \qquad (1)$$

$$e^t v_1' - (2t+2)v_2' = t^2 \qquad (2)$$

Subtract: (2) - (1):

$$-(2t+2)v_2' + (t^2+2t+2)v_2' = +t^2$$

$$t^2 v_2' = +t^2 \Rightarrow v_2' = +1 \Rightarrow v_2 = +t$$

Substitute into (1):

$$v_1' = (t^2+2t+2)e^{-t} \Rightarrow v_1 = -e^{-t}(t^2+2t+2) - e^{-t}(2t+2) - 2e^{-t}$$

So, $y_p = -(t^2+2t+2) - (2t+2) - 2 + t(-t^2-2t-2)$

$$= -t^3 - 3t^2 - 6t - 6$$

g) General solution:

$$y(t) = c_1 y_1 + c_2 y_2 + y_p = c_1 e^t + c_2(t^2+2t+2) - t^3 - 3t^2 - 6t - 6$$

8) $\ddot{X} + 2\dot{X} + X = 25\sin(+2t)$

a) 1N required to displace 1m, so $k = \frac{1N}{1m} = 1\,N/m$

$$m\ddot{X} + \overset{b}{\cancel{2}}\dot{X} + kX = 25\sin(-2t) \Rightarrow m = 1\text{ kg},\quad b = 2\,\frac{N\cdot s}{m}$$

b) Over/under damped?

$$r^2 + 2r + 1 \qquad r = \frac{-2 \pm \sqrt{4-4}}{2} = -1 \leftarrow \text{critically damped}$$

c) $X_h = c_1 e^{-t} + c_2 t e^{-t}$

Use MUC: $X_p = A\cos(2t) + B\sin(2t)$

$$\dot{X}_p = -2A\sin(2t) + 2B\cos(2t)$$

$$\ddot{X}_p = -4A\cos(2t) - 4B\sin(2t)$$

So, $-4A\cos(2t) - 4B\sin(2t) - 4A\sin(2t) + 4B\cos(2t) + A\cos(2t) + B(\sin(2t)) = 25\sin(2t)$

$$\Rightarrow \begin{aligned} -4A + 4B + A &= 0 \\ -4B - 4A + B &= 25 \end{aligned} \quad \rightarrow \quad \begin{aligned} -3A + 4B &= 0 \\ -4A - 3B &= 25 \end{aligned}$$

Has soln $A = -4$, $B = -3$, so

$$X_p = -4\cos(2t) - 3\sin(2t)$$

$$X_g(t) = c_1 e^{-t} + c_2 t e^{-t} - 4\cos(2t) - 3\sin(2t)$$

d) $X(0) = 0$, $\dot{X}(0) = 1$.

$$0 = c_1 - 4 \Rightarrow c_1 = 4$$

$$\dot{X} = -4e^{-t} + c_2 e^{-t} - c_2 t e^{-t} + 8\sin(2t) - 6\cos(2t)$$

$$1 = -4 + c_2 - 6 \Rightarrow c_2 = 11$$

$$X(t) = 4e^{-t} + 11te^{-t} - 4\cos(2t) - 3\sin(2t)$$

e) System oscillates for all time. Stays bounded.

f) Write homo. eqn as linear system

$$\begin{aligned} X_1 &= X \\ X_2 &= \dot{X} \end{aligned} \Rightarrow \ddot{X} = \dot{X}_2 \Rightarrow \dot{X}_2 + 2X_2 + X_1 = 0 \qquad \begin{aligned} \dot{X}_1 &= X_2 \\ \dot{X}_2 &= -X_1 - 2X_2 \end{aligned} \qquad \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

9) $A = \begin{pmatrix} 1 & t & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

a) E-vals are $\lambda_1 = 1$, $\lambda_2 = 2$ $\quad P_A(\lambda) = (1-\lambda)^2(2-\lambda)$

λ_1 has alg. mult. 2, λ_2 has alg. mult 1

b) $\underline{\lambda = 1}$

$$\left[\begin{array}{ccc|c} 0 & t & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

i) Always has soln, E_1 never 0-D

ii) If $t \neq 0$:

$x_1 = s \quad tx_2 = 0 \Rightarrow x_2 = 0 \quad x_3 = 0$, so $v_1 = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow$ 1-D

iii) If $t = 0$

$x_1 = s$

$x_2 = r \Rightarrow u_1 = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \leftarrow$ 2-D

$x_3 = 0$

iv) Never 3-D

c) If $t=1$, then $\lambda = 1$ has geo. mult. 1 since E_1 is 1-D. Since $\lambda = 2$ has alg. mult. 1, E_2 is always 1-D.

10) $y'' + 6y' + 9y = 0 \quad y_h = c_1 e^{-3t} + c_2 t e^{-3t}$

a) $t(1-t) \quad y_p = At^2 + Bt + C$

b) $e^{-3t} \quad y_p = At^2 e^{-3t}$

$y'' + 4y = 0 \quad y_h = c_1 \cos(2t) + c_2 \sin(2t)$

c) $\sin(t) \quad y_p = A\cos(t) + B\sin(t)$

d) $y'' = 0 \quad y_h = c_1 + c_2 t$

$t(1-t) \quad y_p = At^4 + Bt^3 + Ct^2$

$y'' + 2y' - 3y = 0 \quad y_h = c_1 e^{t} + c_2 e^{-3t}$

e) $t^2 - 3 \quad y_p = At^2 + Bt + C$

f) e^{t^2} NOT

g) $\sin(t) + \cos(5t)$ NOT

h) $t^3 e^{-3t} \quad y_p = (At^4 + Bt^3 + Ct^2 + Dt)e^{-3t}$

i) $4e^{-3t}\sin(t) \quad y_p = (A\cos(t) + B\sin(t))e^{-3t}$

11) Solve $\underline{x}' = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} \underline{x}$

1 eigenvalue, $\lambda = 4$

1 eigenvector $\underline{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Need generalized eigenvector. Solve $(A - \lambda I)\underline{u} = \underline{v}$.

Has solution $\underline{u} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \underset{=}{\overset{s=1}{=}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Then $\left(\underline{x} = c_1 e^{\overset{\lambda}{\cancel{4}}t} \underline{v} + c_2 e^{\lambda t}(t\underline{v} + \underline{u}) \right)$

$$\underline{x}(t) = c_1 e^{4t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{4t} \left(t \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$= c_1 e^{4t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} -t \\ 2t+1 \end{pmatrix}$$

12) i) $x' = 4x - 5y$, $y' = 5x - 4y$ $\quad \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix}$ has evals $\pm 3i \Rightarrow$ center (a)

ii) $x' = x + y$, $y' = 4x + y$ $\quad \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ has evals $3, -1 \Rightarrow$ saddle (b)

iii) $x' = 2x + y$, $y' = 6x + 3y$ $\quad \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$ has evals $0, 5 \Rightarrow$ unstable (c)

iv) $x' = 2x - y$, $y' = 4x + 6y$ $\quad \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}$ has double ~~evals~~ eval $4 \Rightarrow$ unstable (d)

13) $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

a) $\det(A) = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} = (2-1) + (-1) = 0$

b) Evals + Evecs:

$$\lambda_1 = 3, \quad \underline{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad \lambda_2 = 1 \quad \underline{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \lambda_3 = 0 \quad \underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

c) IVP

$$\begin{aligned} u' &= u - v \\ v' &= -u + 2v - w \\ w' &= -v + w \end{aligned} \rightsquigarrow \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

General soln:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

IC: $\begin{pmatrix} u(0) \\ v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ so:

$$\begin{aligned} c_1 - c_2 + c_3 &= 2 \\ -2c_1 \quad + c_3 &= -1 \\ c_1 + c_2 + c_3 &= 2 \end{aligned}$$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ -2 & 0 & 1 & -1 \\ 1 & 1 & 1 & 2 \end{array}\right]$ has solution $c_1 = 1$, $c_2 = 0$, $c_3 = 1$

d) Basis $= \left\{ e^{3t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, e^{t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

e) $\begin{pmatrix} 1 & 0 & 1 \\ -\cancel{2}1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ Evals $= 2, 0$ double root

$\lambda_1 = 2, \quad \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ← degenerate eigenspace

$\lambda_2 = 0, \quad \underline{v}_2 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$

14) a) Suppose $|A-\lambda I|=0$. Then

$|B^{-1}AB-\lambda I| = |B^{-1}AB-\lambda B^{-1}B| = |B^{-1}(A-\lambda I)B| = |B^{-1}|\,|A-\lambda I|\,|B|$

$=0 \Rightarrow \lambda$ is eval of $B^{-1}AB$

b) Let $\lambda, \underline{u}$ satisfy $A\underline{u}=\lambda\underline{u}$. Then

$(A-cI)\underline{u} = A\underline{u}-c\underline{u} = \lambda\underline{u}-c\underline{u} = (\lambda-c)\underline{u}$,

so $\underline{u}$ is evec of $A-cI$ w/ eval $\lambda-c$

c) 1^5 and 2^5

d) 2. $\underline{v}_1$ and $\underline{v}_2$ are linearly independent by distinct eigenvalue thm.

e) i) False

ii) True by eigenspace thm

iii) True by definition of eval

f) i) False, we need that λ_1 and λ_2 are complex conjugates

ii) True for example the identity matrix

iii) False, at least one must be purely real

15) $x' = y$

$y' = -y + x - x^3$

a) Equilibria:

$$\begin{aligned} 0 &= y \\ 0 &= -y + x - x^3 \end{aligned} \rightarrow \begin{aligned} y &= 0 \\ x &= x^3 \end{aligned} \rightarrow \begin{aligned} y &= 0 \\ x &= \pm 1, 0 \end{aligned}, \quad (-1,0), (0,0), (1,0)$$

$$J = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & -1 \end{pmatrix}$$

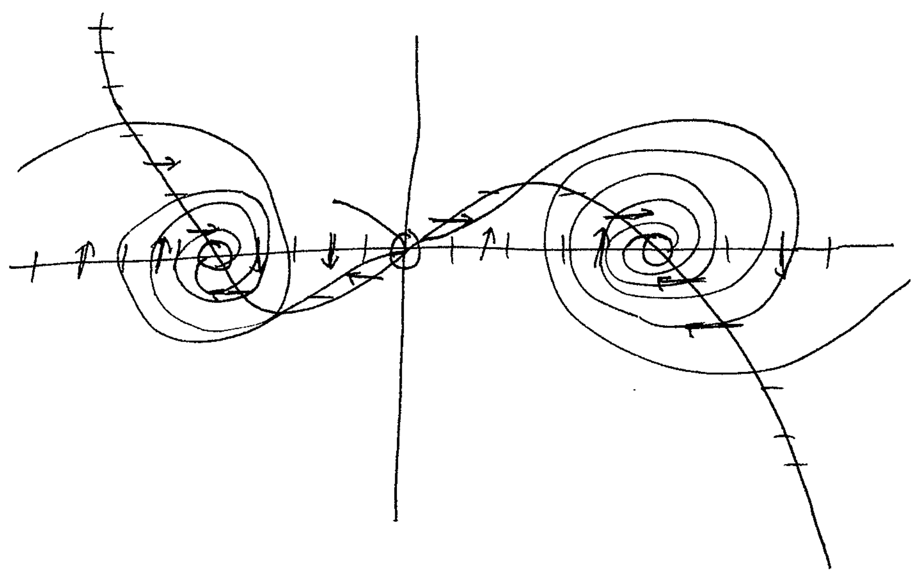
At $(0,0)$, $J(0,0)$ has evals $-\frac{1}{2} \pm \frac{\sqrt{5}}{2} \Rightarrow$ <u>saddle</u>

At $(-1,0)$, $J(-1,0)$ has evals $-\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \Rightarrow$ <u>stable spiral</u>

At $(1,0)$, $J(1,0)$ has evals $-\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \Rightarrow$ <u>stable spiral</u>

b) Vertical nullclines at $y = 0$

Horizontal nullclines at $y = x - x^3$



16) a) Decay problem: ~~$t=0, Q(t) \Rightarrow$~~

$$\frac{dQ}{dt} = kQ \qquad Q(0) = 100$$
$$Q(50) = 75$$

General soln: $Q(t) = ce^{kt}$

$$Q(0) = 100 = ce^{k0} \Rightarrow c = 100$$

$$Q(50) = 75 = 100e^{50k} \Rightarrow k = \frac{\ln(3/4)}{50}$$

$$Q(t) = 100 \exp\left[\frac{\ln(3/4)}{50} t\right]$$

Half-life?

$$\frac{1}{2} = \exp\left[\frac{\ln 3/4}{50} t_h\right] \Rightarrow t_h = \frac{\ln 1/2}{\ln 3/4} \cdot 50$$

b) Mixing problem. X is lb of salt in tank.

$$X' = \text{Rate In} - \text{Rate Out} = \frac{1\text{lb}}{\text{gal}} \cdot \frac{3\text{ gal}}{\text{min}} - \frac{X}{V} \cdot \frac{1\text{ gal}}{\text{min}} = 3 - \frac{X}{300+2t}$$

$$\xrightarrow{IF} X(t) = (300+2t) + c(300+2t)^{-1/2}$$

$$X(0) = 0 = 300 + \frac{c}{\sqrt{300}} \Rightarrow c = -300\sqrt{3 \cdot 100} = -3000\sqrt{3}$$

So

$$X(t) = (300+2t) - 3000\sqrt{3}(300+2t)^{-1/2}$$

Tank is full when $V(t_f) = 600 = 300 + 2t_f$, so $t_f = 150$

Then

$$X(150) = 600 - 3000\sqrt{3}(600)^{-1/2}$$

c) Cooling problem.

$$T' = k(M-T) \Rightarrow T(t) = T_0 e^{-kt} + M(1 - e^{-kt})$$

$$T_0 = 200^\circ;\ M = 70^\circ$$

Solve for k:

$$T(15) = 120 = 200e^{-15k} + 70 - 70e^{-15k} = 70 + 130e^{-15k}$$

$$\Rightarrow k = \frac{-\ln\frac{5}{13}}{15}$$

$$\therefore T(t) = 200\exp\left[\frac{\ln\frac{5}{13}}{15}t\right] + \cancel{200}\ 70\left(1 - \exp\left[\frac{\ln\frac{5}{13}}{15}t\right]\right)$$