DIFFERENTIATION

Basic rules

In order to differentiate a function there are some basic rules that we use almost all the time. They can of course be derived, but it would be tedious to start from scratch for each differentiation, so it is better to know them.

Powers of x whether n is an integer or not follows the rule

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Ex. 1 $\frac{d}{dx}1 = 0$

Ex. 2 $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

Ex. 3 $\frac{d}{dx}x^{-3} = -3x^{-4}$

We need to know the derivatives of elementary functions such as

Ex. 1 $\frac{d}{dx}\sin x = \cos x$

Ex. 2 $\frac{d}{dx}\cos x = -\sin x$

Ex. 3 $\frac{d}{dx}e^x = e^x$

Ex. 4 $\frac{d}{dx}\ln x = \frac{1}{x}$

Ex. 5 $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$

Ex. 6 $\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$

Ex. 7 $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

The product rule and the quotient rule (follows from the product rule) are also very useful

$$\frac{d}{dx}u(x)v(x) = u'(x)v(x) + u(x)v'(x),$$

$$\frac{d}{dx}\frac{u(x)}{v(x)} = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}.$$

The chain rule

Assuming that we know how to differentiate simple expressions, the most important tool for differentiation of more complex things is the chain rule. Consider the function

$$f(x) = \sin(\sqrt{x^3 - 2x^2 + 1}),$$

that we would like to differentiate. Let $u(x) = \sqrt{x^3 - 2x^2 + 1}$. We can easily compute

$$\frac{df}{du} = \frac{d}{du}(\sin u) = \cos u.$$

Then let $s(x) = x^3 - 2x^2 + 1$, which gives us

$$\frac{du}{ds} = \frac{d}{ds}\left(\sqrt{s}\right) = \frac{1}{2\sqrt{s}}.$$

Now the expression is so simple that we can also compute

$$\frac{ds}{dx} = \frac{d}{dx}\left(x^3 - 2x^2 + 1\right) = 3x^2 - 4x.$$

The chain rule tells us that we can use these steps to find the derivative we wanted in the following way:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{ds}\frac{ds}{dx}.$$

We can insert any number of steps we like to break up the differentiation into simple parts. For the example above we get

$$\frac{df(x)}{dx} = \cos u \frac{1}{2\sqrt{s}}(3x^2 - 4x) = \frac{1}{2}\frac{\cos(\sqrt{x^3 - 2x^2 + 1})(3x^2 - 4x)}{\sqrt{x^3 - 2x^2 + 1}}$$

Example: Compute $f'(x)$ when $f(x) = \exp((1 + x^2)^4)$.

$$f(x) = e^{(1+x^2)^4} = e^u$$
$$u(x) = (1 + x^2)^4 = s^4$$
$$s(x) = (1 + x^2)$$

The chain rule gives that

$$f'(x) = \frac{df}{du}\frac{du}{ds}\frac{ds}{dx} = e^u 4s^3 2x = 8e^{(1+x^2)^4}x(1 + x^2)^3$$

Note 1: This can also be thought of as "taking the inner derivative".
Note 2: We can look at the chain rule as if we are inserting for example du both in the numerator and the denominator, which makes no difference in the final result.

Taylor expansion is not used for differentiation but it uses derivatives to represent functions as polynomials. Any differentiable function can be expressed as a Taylor expansion around a point x_0 in the following way

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)} + \cdots .$$

Usually the Taylor expansion is used for approximating the function close to x_0 by only using a small number of terms from the series. The linear approximation is

$$T_1(x) = f(x_0) + (x - x_0)f'(x_0).$$

Using $T_1(x)$ as an approximation is saying that close to x_0 the function is well approximated by its tangent through the point $(x_0, f(x_0))$.

The accuracy and the interval in which the approximation is reasonable increases with the number of terms used.

Example: Find an approximation for $f(x) = \sin(x)$ for small values of x. Expand the function around $x_0 = 0$, since we are considering small x. Then we need the derivatives of $\sin(x)$ evaluated at 0.

$$\begin{array}{lllllllll}
f(0) & = & f^{(4)}(0) & = & f^{(4m)}(0) & = & \sin(0) & = & 0 \\
f'(0) & = & f^{(5)}(0) & = & f^{(4m+1)}(0) & = & \cos(0) & = & 1 \\
f''(0) & = & f^{(6)}(0) & = & f^{(4m+2)}(0) & = & -\sin(0) & = & 0 \\
f^{(3)}(0) & = & f^{(7)}(0) & = & f^{(4m+3)}(0) & = & -\cos(0) & = & -1
\end{array}$$

The first approximations then are

$$\begin{array}{lllll}
T_1(x) & = & 0 + 1(x - 0) & = & x \\
T_3(x) & = & x + 0\frac{(x-0)^2}{2} - 1\frac{(x-0)^3}{3!} & = & x - \frac{x^3}{6} \\
T_5(x) & = & x - \frac{x^3}{6} + 0\frac{(x-0)^4}{4!} + 1\frac{(x-0)^5}{5!} & = & x - \frac{x^3}{6} + \frac{x^5}{120}
\end{array}$$

The figure shows how the approximations close in on the true sine function.