INTEGRATION TECHNIQUES

Introduction

We are looking for ways to compute the integral of a function $f(x)$,

$$\int f(x)dx.$$

To put it simply, what we need to do is find a function $F(x)$ such that $F'(x) = f(x)$. Then if the integral is definite, that is, we have an interval over which to integrate, the solution is

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a).$$

For an indefinite integral (no interval) the solution is

$$\int f(x)\,dx = \int F'(x)\,dx = F(x) + c,$$

where c is an arbitrary constant. The integrals that arise in differential equation applications are mostly of the latter type and there the constant c plays a crucial part and should not be forgotten.

Selected special cases

Powers of x (possibly including a constant) are straightforward to integrate. The following formula works for both integer and noninteger values of n.

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1} + c, \quad n \neq -1, \quad (a = 0 \text{ when } n = 0).$$

Ex. 1 $\int 1\,dx = x + c$

Ex. 2 $\int x^{3.2}\,dx = \frac{x^{4.2}}{4.2} + c$

Ex. 3 $\int (x^4 + 3x^3 - 2x^2 + x - 1)\,dx = \frac{x^5}{5} + 3\frac{x^4}{4} - 2\frac{x^3}{3} + \frac{x^2}{2} - x + c$

Ex. 4 $\int (x+3)^4 dx = \frac{(x+3)^5}{5} + c$

Ex. 5 $\int (2x+3)^4 dx = \frac{1}{2}\frac{(2x+3)^5}{5} + c$ (Note the extra coefficient.)

Ex. 6 $\int \frac{1}{\sqrt{x-1}} dx = \int (x-1)^{-\frac{1}{2}} dx = 2(x-1)^{\frac{1}{2}} + c$

The derivative of the logarithm has a special appearance, which often makes it easy to spot in an integral. The formula looks like this:

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c.$$

The absolute value in the logarithm is there for a reason. Both $\ln f(x)$ and $\ln(-f(x))$ are solutions, but only the one with a positive value gives a well defined answer. Taking the absolute value chooses the right solution.

Ex. 1	$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + c$
Ex. 2	$\int \frac{x}{3-x^2} dx = -\frac{1}{2} \int \frac{-2x}{3-x^2} dx = -\frac{1}{2} \ln|3-x^2| + c$
Ex. 3	$\int \frac{1}{x+1} dx = \ln|x+1| + c$

The functions e^{x^2} and e^{-x^2} are a bit special. It is necessary to know what can and cannot be done, and also to avoid confusing it with similar functions.

Ex. 1	$\int 2xe^{x^2} \, dx = e^{x^2} + c$
Ex. 2	$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$
Ex. 3	$\int e^{x^2} dx$ No closed form solution exists.
Ex. 4	$\int (e^x)^2 \, dx = \int e^{2x} dx = \frac{e^{2x}}{2} + c$

Integration by parts

Integration by parts is especially useful when the integrand has two parts that are not too complicated, but together form something that is hard to integrate. The basis for the method is the product rule for derivatives:

$$\frac{d}{dx}(u(x)v(x)) = u'(x)v(x) + u(x)v'(x).$$

Integrating both sides with respect to x and moving the terms around yields

$$\int u'v \, dx = uv - \int uv' dx.$$

In the first example, we use this for a cosine or sine function multiplied by an exponential function.

$$\int \underbrace{e^{\alpha x}}_{u'} \underbrace{\cos\beta x}_{v}\, dx = \underbrace{\frac{1}{\alpha}e^{\alpha x}}_{u} \underbrace{\cos\beta x}_{v} - \int \underbrace{\frac{1}{\alpha}e^{\alpha x}}_{u} \underbrace{(-\beta\sin\beta x)}_{v'}\, dx \tag{1}$$

$$\int \underbrace{e^{\alpha x}}_{u'} \underbrace{\sin\beta x}_{v}\, dx = \underbrace{\frac{1}{\alpha}e^{\alpha x}}_{u} \underbrace{\sin\beta x}_{v} - \int \underbrace{\frac{1}{\alpha}e^{\alpha x}}_{u} \underbrace{(\beta\cos\beta x)}_{v'}\, dx \tag{2}$$

Using (2) to replace the integral in the right hand side of (1) gives

$$\int e^{\alpha x}\cos\beta x\, dx = \frac{1}{\alpha}e^{\alpha x}\cos\beta x + \frac{\beta}{\alpha}\left(\frac{1}{\alpha}e^{\alpha x}\sin\beta x - \int \frac{1}{\alpha}e^{\alpha x}\beta\cos\beta x\, dx\right).$$

By rearranging the equation and adding the constant c we get the final result

$$\int e^{\alpha x}\cos\beta x\, dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\alpha\cos\beta x + \beta\sin\beta x\right) + c.$$

The integral for the sine function can be found in a very similar way.

Here is an example that is less tricky, but requires several steps of integration by parts (u and v are different functions in each step).

$$\begin{aligned}
\int \underbrace{x^3}_{u} \underbrace{e^{ax}}_{v'}\, dx &= \underbrace{x^3}_{u} \underbrace{\frac{e^{ax}}{a}}_{v} - \int \underbrace{3x^2}_{u'} \underbrace{\frac{e^{ax}}{a}}_{v}\, dx \\
&= x^3\frac{e^{ax}}{a} - \underbrace{3x^2}_{u} \underbrace{\frac{e^{ax}}{a^2}}_{v} + \int \underbrace{6x}_{u'} \underbrace{\frac{e^{ax}}{a^2}}_{v} \\
&= x^3\frac{e^{ax}}{a} - 3x^2\frac{e^{ax}}{a^2} + \underbrace{6x}_{u} \underbrace{\frac{e^{ax}}{a^3}}_{v} - \int \underbrace{6}_{u'} \underbrace{\frac{e^{ax}}{a^3}}_{v} \\
&= x^3\frac{e^{ax}}{a} - 3x^2\frac{e^{ax}}{a^2} + 6x\frac{e^{ax}}{a^3} - 6\frac{e^{ax}}{a^4} + c
\end{aligned}$$

Integration by parts does not always help. The integral of a rational function,

$$\int \frac{p_{n-1}(x)}{p_n(x)} dx,$$

where $p_{n-1}(x)$ and $p_n(x)$ are polynomials of degree $n-1$ and n, can be computed using partial fractions.

It is usually fairly easy to put an expression on a common denominator, for example,

$$\begin{aligned}\frac{3x+1}{x^2+4} + \frac{1}{x-2} - \frac{1}{x+2} &= \frac{(3x+1)(x-2)(x+2) + (x^2+4)(x+2) - (x^2+4)(x-2)}{(x^2+4)(x-2)(x+2)} \\ &= \frac{3x^3+5x^2-12x+12}{x^4-16} = q(x). \qquad (3)\end{aligned}$$

Partial fractions is the art of doing the same thing backwards. The procedure consists of three steps, here illustrated for the fraction $q(x)$ in (3).

1. Find the factors in the denominator (almost the same as finding the roots):

   $$(x^4-16) = (x^2+4)(x^2-4) = (x^2+4)(x+2)(x-2)$$

   For a polynomial with real coefficients, all factors will be first or second order polynomials.

2. Write down the form of the partial fractions. Each second order denominator gets a first order numerator and each first order denominator gets a constant numerator.

   $$\frac{Ax+B}{x^2+4} + \frac{C}{x+2} + \frac{D}{x-2},$$

   where A, B, C, and D are unknown constants.

3. Determine the constants so that the sum of partial fractions equals the original fraction $q(x)$. This can be done in the following way. Put the fractions with constants back on a common denominator and then set the old and new numerators to be equal:

   $$(Ax+B)(x^2-4)+C(x^2+4)(x-2)+D(x^2+4)(x+2) = 3x^3+5x^2-12x+12.$$

   Choose a number of "good" x-values to get conditions for the constants.

   $$\begin{aligned} x=-2 &\Rightarrow & -32C = 32 &\Leftrightarrow C=-1 \\ x=2 &\Rightarrow & 32D = 32 &\Leftrightarrow D=1 \\ x=0 &\Rightarrow & -4B-8C+8D = 12 &\Rightarrow B=1 \\ x=1 &\Rightarrow & -3A-3B-5C+15D = 8 &\Rightarrow A=3 \end{aligned}$$

Now, when the original expression is decomposed into partial fractions it is possible to perform the integration. In our case, we get

$$\begin{aligned}
\int q(x)dx &= \int \frac{3x+1}{x^2+4} + \frac{1}{x-2} - \frac{1}{x+2}dx \\
&= \frac{3}{2}\int \frac{2x}{x^2+4}dx + \frac{1}{2}\int \frac{1/2}{(x/2)^2+1}dx + \int \frac{1}{x-2}dx - \int \frac{1}{x+2}dx \\
&= \frac{3}{2}\ln(x^2+4) + \frac{1}{2}\arctan\frac{x}{2} + \ln|x-2| - \ln|x+2| + c
\end{aligned}$$

Note 1: If the numerator in the rational function is of a higher degree than the denominator, polynomial division should be carried out first. For example:

$$\frac{x^5+2x^4+3x^3-x^2+x+1}{x^3-1} = x^2+2x+3+\frac{3x+4}{x^3-1}$$

Note 2: A factor that is repeated more than once in the denominator gives special terms:

$$\begin{aligned}
\text{A factor } (x-3)^n &\Rightarrow \frac{A_1}{(x-3)} + \frac{A_2}{(x-3)^2} + \cdots + \frac{A_n}{(x-3)^n}. \\
\text{A factor } (x^2+4)^n &\Rightarrow \frac{A_1x+B_1}{(x^2+4)} + \frac{A_2x+B_2}{(x^2+4)^2} + \cdots + \frac{A_nx+B_n}{(x^2+4)^n}.
\end{aligned}$$