APPM 2360 — Exam final May 6, 1995.

On the front of your bluebook, write your instructor's name, your name, and student number.

There are TEN questions. You must work all problems. Show **all** your work in your bluebook and box in your answers. Calculators and books are NOT permitted. No crib sheet.

1. (20 points) Given $\frac{dx}{dt} = tx + t$
   a) Find the general solution $x(t)$.
   b) Find the solution that satisfies $x(0) = 1$.
   c) Over what interval is the solution from part b) valid?

2. (20 points) Given
$$\frac{dy}{dx} = 4x^3(3y^2 - 4)^{-1}.$$
   a) Find the general implicit solution.
   b) Determine the implicit solution which passes through $(x_0, y_0) = (1, 0)$.

3. (20 points) The differential equation of an unforced simple harmonic oscillator is
$$mu'' + \gamma u' + ku = 0,$$
   where $u(t)$ is the displacement, m is the mass, γ is the friction constant, and k is the spring constant. Suppose a mass of 100 grams is attached to a spring with constant $k = 14400$dyne/cm. Suppose the mass is released , with no initial velocity, +5cm from the equilibrium position. There is no air resistance.
   a) Determine the differential equation for the displacement and the corresponding initial conditions.
   b) Find a general solution for the displacement as a function of time.
   c) Determine the displacement of the mass at any time t by using the initial conditions.

4. (20 points) Given $y'' - y' = 2 - x$
   a) Find two linearly independent solutions of the associated homogeneous equation, and verify the independence with the Wronskian.
   b) Find the general solution of the associated homogeneous equation.
   c) Find a particular solution by the method of undetermined coefficients.
   d) Find a particular solution by the method of variation of parameters

5. (20 points) Given the fourth order ODE
$$y^{(iv)} - 5y'' + 4y = 0$$
   a) Let $x_1 = y$, $x_2 = y'$, $x_3 = y''$, $x_4 = y'''$, and convert the equation to a four-dimensional first-order coupled system.
   b) Write the system as $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$. State explicitly $A =$ _____?
   c) Find the characteristic polynomial of A.

6. (20 points) Given the following set of equations

$$\begin{cases} x_1 + 3x_2 + x_3 = 0 \\ 2x_1 + 4x_2 - x_3 = 0 \end{cases}$$

a) Write the above system in matrix-vector form, $A\mathbf{x} = \mathbf{0}$.
b) Solve the system and write the solution in vector form.
c) Find an integer solution for x_1, x_2, and x_3.

7. (20 points) Given $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$.

a) Find the characteristic polynomial and eigenvalues.
b) Find a fundamental set of real vector solutions.
c) Find a real fundamental matrix solution $X(t)$.
d) What are the critical point(s)?
e) Determine the stability of the critical point(s).

8. (20 points) Given $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$, the general solution is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

a) Write the solution in the form of $\mathbf{x}(t) = X(t)\mathbf{c}$.
b) Find $X^{-1}(0)$.
c) Find e^{tA}.

9. (20 points) Given the equation $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

a) What is the characteristic polynomial ?
b) Determine the eigenvalue(s) with multiplicity.
c) For each eigenvalue, find all possible eigenvectors.
d) Find a fundamental set of solutions.

10. (20 points) For each of the following statements, write down "TRUE" or "FALSE", depending on whether the statements is true or false.

a) $y(t) = t^{-1}$ is a solution of $2t^2y'' + 3ty' - y = 0$, $t > 0$.
b) A homogeneous system $A\mathbf{x} = \mathbf{0}$ where A is $m \times n$ and $n > m$ has only trivial solutions.
c) A homogeneous system $A\mathbf{x} = \mathbf{0}$ where A is $m \times n$ and $n < m$ has only trivial solutions.
d) The solution of $\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$ is given by $\mathbf{x}(t) = X(t)X^{-1}(t_0)\mathbf{x}_0$ where $X(t)$ is any fundamental matrix solution of $X' = AX$.
e) If $detA \neq 0, \quad A\mathbf{x} = \mathbf{b}$ has exactly one solution.
f) If $rankA = rank(A|b)$, then $A\mathbf{x} = \mathbf{b}$ has at least one solution and possibly infinitely many.
g) If $A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 3 \end{pmatrix}$, then every solution of $\mathbf{x}' = A\mathbf{x}$ tends to $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ as $t \to \infty$.