APPM 2360—Exam final **December 15, 1995**

On the front of your bluebook, write your name, student #, and Lecturer's name.

There are TEN problems. You must work all problems. Show **all** your work in your bluebook and box in your final answers. Calculators and books are NOT permitted. No crib sheets are allowed.

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1. (20pts) Given $y' = y^2 \sin x$
   (a) Find the general solution $y(x)$.
   (b) Find the solution $y(x)$ which satisfies $y(0) = 1$.

2. (20pts) Given $y' - 2xy = e^{x^2}$
   (a) Find the general solution $y(x)$.
   (b) Find the solution $y(x)$ which satisfies $y(0) = -1$.

3. (20pts) Given $y'' - 3y' - 4y = 16x$
   (a) Write down the corresponding homogeneous equation and find a general solution of this homogeneous equation.
   (b) Find a particular solution of the non-homogeneous equation.
   (c) Find the general solution of the non-homogeneous equation.

4. (20pts) Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

   (a) Find the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
   (b) Find all eigenvalues of A.
   (c) Is A invertible?
   (d) Find a basis of the eigenvectors (i.e., a set of linearly independent eigenvectors) corresponding to the smallest eigenvalue of A.

5. (20pts) For each of the following, answer TRUE or FALSE.
   (a) If $\det A = 0$, then $Ax = 0$ has infinitely many solutions.
   (b) If $y(t)$ and $y_p(t)$ are solutions of $x' = Ax + f(t)$, then $y(t) - y_p(t)$ is a solution of $x' = Ax$.
   (c) A homogeneous system $Ax = 0$ where A is $m \times n$ and $n < m$ has only trivial solutions.
   (d) Three vectors in $\mathbb{R}^4$ are always linearly independent.
   (e) For $x' = \begin{bmatrix} -1 & 4 \\ -4 & -1 \end{bmatrix} x$, the equilibrium solution (i.e. critical point) is asymptotically stable.

6. (20pts) Given $x' = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix} x$

   (a) Find the eigenvalues.
   (b) Find the general solution.
   (c) Find a fundamental matrix solution $\Psi(t)$.

7. (20pts) Given $y^{(iv)} + y''' - 5y'' - y' = 0$.

   (a) Let $x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad x_4 = y'''$ and convert the given equation into a system of 4 coupled first order differential equations.
   (b) Write the system in a) in matrix-vector form $x' = Ax$.
   (c) Find the characteristic polynomial.

8. (20pts) Given $x' = Ax$ where $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{bmatrix}$, the general solution is given by

$$x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

   (a) Write the solution in the form $x(t) = \Psi(t) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.
   (b) Find $\Psi^{-1}(0)$.
   (c) Find e^{tA}.
   (d) Find $(e^{tA})^{-1}$.

9. (20pts) Given $x' = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} x + \begin{pmatrix} 3e^{3t} \\ e^{3t} \end{pmatrix}$

   (a) Find the fundamental matrix Φ with $\Phi(0) = I$ to the corresponding homogeneous equation.
   (b) Find a particular solution to the non-homogeneous problem.
   (c) Find the general solution to the non-homogeneous problem, and solve the initial value problem $x(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

10. (20pts) For the following equations a) – d), write down the corresponding phase portrait from the ones shown in (i) – (viii), and state whether the equilibrium of the equation is stable or unstable.

a) $x' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} x$ b) $x' = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix} x$

c) $x' = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} x$ d) $x' = \begin{bmatrix} 7 & -3 \\ 18 & -8 \end{bmatrix} x$