Final Exam

APPM 2360, Spring 1996
May 15, 1996

ON THE FRONT OF YOUR BLUEBOOK write your name, student #, (3) lecture section (020 for 12-1 PM, and 030 for 2-3 PM, 040 for 8-9 AM).

There are TEN questions. You must work all of the problems. Show ALL of your work in your bluebook and box in your final answers. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Start each problem on the top of a new page. You may use a calculator. You may have no other papers or books or devices of any sort (nuclear bombs, computers, etc.) on your desk.

1) (20 pts) Solve the initial value problems

a) $\frac{dy}{dt} = \frac{\cos t}{2 + y}$, $y(0) = 0$

b) $\frac{y'}{x} = y^3$, $y(1) = -2$

2)(20 pts) Find the general solution of the following differential equations

a) $\frac{dy}{dx} + \frac{1}{x}y = e^{x^2}$

b) $y'' - 2y' + y = e^x$

3) (20 pts) Consider the differential equation

$$y'' + y = \frac{1}{2}\sec x,\ y(0) = 0,\ y'(0) = 1$$

a) According to the theorem on existence and uniqueness, on what interval of x is the solution guaranteed to exist and be unique?
b) Find the solution of the equation.
c) On what interval of x does the solution exist?

4) (20 pts) A tank contains 10,000 *liters* of a solution consisting of 100 *kg* of salt dissolved in water. Pure rocky mountain spring water is pumped into the tank at the rate of 10 *liters/second*. (The mixture is kept uniform by stirring). The mixture is pumped out of the tank at the same rate it enters.

a) Set up the initial value problem for the amount of salt, $S(t)$, in the tank at time t.
b) Solve this problem for $S(t)$.
c) How long will it be before only 50 *kg* of salt remain in the tank?

5) (20 pts) Indicate in your blue book if the following are TRUE or FALSE.

a) $y_1 = x^2$ and $y_2 = x^{-1}$ are a fundamental set of solutions of the equation $x^2y'' - 2y = 0$.
b) If **A** is a singular $n \times n$ matrix, then the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution.
c) If $\boldsymbol{A}$ is nonsingular then $\boldsymbol{A}^{-1}$ exists.
d) If $\boldsymbol{A}$ is a 2×3 matrix and $\boldsymbol{B}$ is 3×2 matrix then $\boldsymbol{AB}$ is a 3×3 matrix.
e) If $\boldsymbol{A}$ is a singular $n \times n$ matrix then $\lambda = 0$ is an eigenvalue of $\boldsymbol{A}$.

6) (20 pts) Determine the most general real solution of

$$\frac{dx}{dt} = \begin{pmatrix} 1 & 3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

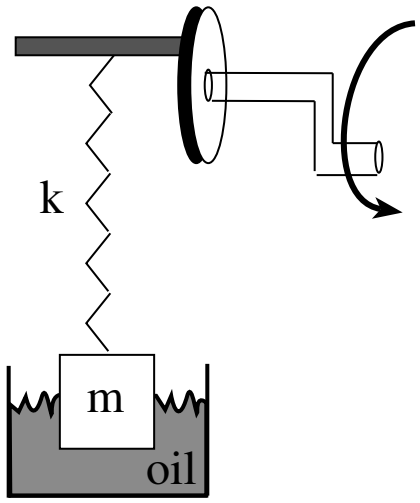
7) (20 pts) Find all values of k for which the following three vectors are
a) Linearly independent,
b) Linearly dependent.

$$\boldsymbol{x}^{(1)} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \boldsymbol{x}^{(2)} = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \quad \boldsymbol{x}^{(3)} = \begin{pmatrix} 1 \\ 1 \\ k \end{pmatrix}$$

c) For both cases (a) and (b), find c_1, c_2, and c_3 so that $c_1\boldsymbol{x}^{(1)} + c_2\boldsymbol{x}^{(2)} + c_3\boldsymbol{x}^{(3)} = \mathbf{0}$.

8) (20 pts) A damped spring is connected to a crank which forces the spring at a frequency of 1/sec. According to the laws of Newton and Hooke, the governing equation is

$$2\ddot{u} + 3\dot{u} + u = 10\sin t$$



a) Explain the physical meaning of each term in the equation
b) Find the general solution of the equation.
c) If the spring is steadily cranked for 2 hours, determine the amplitude of the solution.

9) (20 pts)
a) Find the general solution of the following equation

$$\frac{dx}{dt} = \boldsymbol{A}x \qquad A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

b) Find a fundamental matrix solution, $\Psi(t)$, of this equation.
c) Find $e^{t\boldsymbol{A}}$.

10) (20 pts) For each of the following matrices $\boldsymbol{A}$, consider the system $\frac{dx}{dt} = \boldsymbol{A}x$. In each case, find the eigenvectors, eigenvalues, sketch the phase portrait, and give the stability types (stable, unstable or asymptotically stable).

a) $\boldsymbol{A} = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}$ b) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}$ c) $\boldsymbol{A} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$