APPM 2360 Exam 3 Solutions

1a.

Equation	1	2	3
Relationship	A	C	D

1b. x nullclines occur where $x' = (-2 - x + y)x = 0$

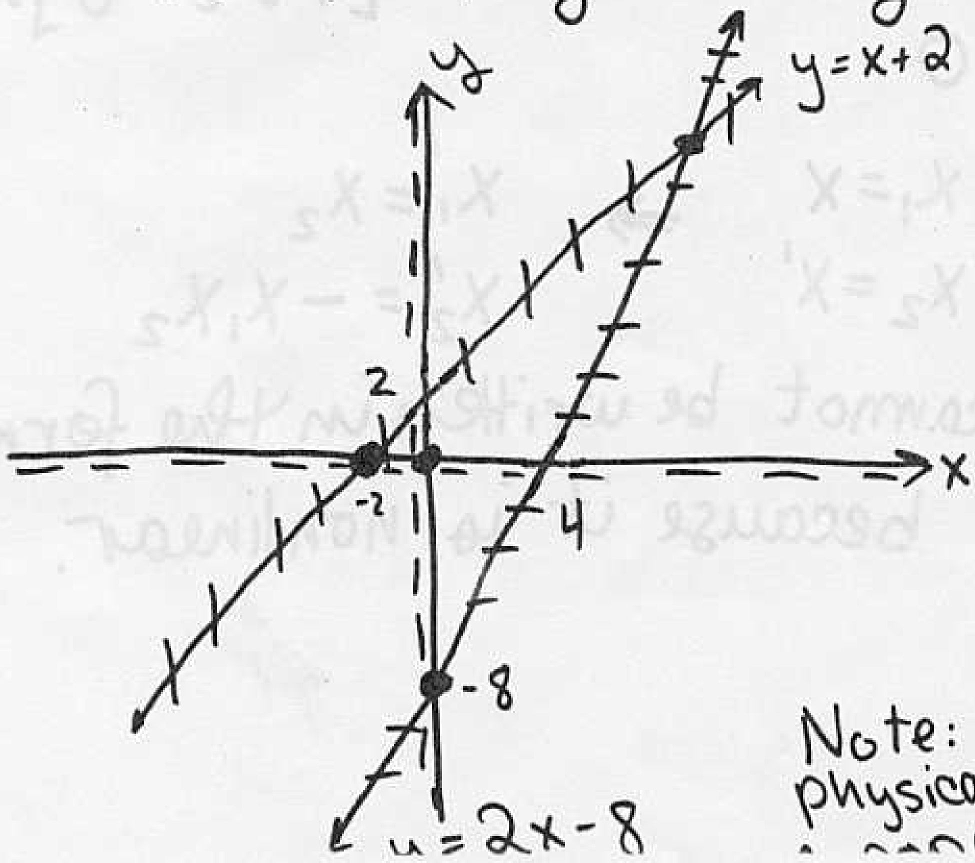
so $x = 0$

or $-2 - x + y = 0 \Rightarrow y = x + 2$

y nullclines occur where $y' = (4 - x + .5y)y = 0$

so $y = 0$

or $4 - x + .5y = 0 \Rightarrow y = 2x - 8$



Equilibrium solutions occur where any x nullcline intersects a y nullcline:

(0,0), (10,12),

(0,-8) & (-2,0).

Note: these last 2 are not physically reasonable for a population problem

2a. $x''' + 4x' + 7x = \cos(t)$

choose $x_1 = x$, $x_2 = x'$, $x_3 = x''$ $\Longrightarrow$ $x_1' = x_2$, $x_2' = x_3$, $x_3' = -7x_1 - 4x_2 + \cos(t)$

Then $x' = Ax + F$ with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -4 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 \\ 0 \\ \cos(t) \end{bmatrix}$

2b. $x'' + t^2 x + e^t x = 4x$

choose $x_1 = x$, $x_2 = x'$ $\Longrightarrow$ $x_1' = x_2$, $x_2' = (4 - t^2 - e^t)x_1$

Then $x' = Ax + F$ with $A = \begin{bmatrix} 0 & 1 \\ 4-t^2-e^t & 0 \end{bmatrix}$, $F = 0$

2c. $x'' + xx' = 0$

choose $x_1 = x$, $x_2 = x'$ $\Longrightarrow$ $x_1' = x_2$, $x_2' = -x_1 x_2$

This system cannot be written in the form $x' = Ax + F$ because it is nonlinear.

3a. Set up the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 2 \end{array}\right]$$

subtract Row(1) and Row(2) from Row(3):

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & -4 \end{array}\right]$$

This leads to the inconsistent equation $0=-4$

Hence there are no solutions to this system of equations.

3b. This system has augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

after subtracting Row(1) & Row(2) from Row(3)

Next subtract Row(1) from Row(2):

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 0 & 3 & 2 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

we have 2 degrees of freedom, so let X_3 and X_4 take on any value

Row(2) $\Rightarrow$ $x_2 = 4/3 - 2/3 x_3 - \frac{5}{3}x_4$

Row(1) $\Rightarrow$ $x_1 = 1 + x_2 - x_3 + x_4$

$$= 1 + \frac{4}{3} - \frac{2}{3}x_3 - \frac{5}{3}x_4 - x_3 + x_4$$

$$= \frac{7}{3} - \frac{5}{3}x_3 - \frac{2}{3}x_4$$

So solutions are given by

$x_1 = \frac{7}{3} - \frac{5}{3}x_3 - \frac{2}{3}x_4$, $x_2 = \frac{4}{3} - \frac{2}{3}x_3 - \frac{5}{3}x_4$

for any values of x_3 , x_4 .

4a. Eigenvalues are roots of $\det(A - \lambda I)$

$$= \det\begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 4 + 2$$

$$= \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$$

$\lambda = 2,\ \lambda = 3$

<u>Eigenspace V_2:</u> Find v s.t. $(A - 2I)v = 0$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} v = 0 \Rightarrow v = s\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any $s \in \mathbb{R}$.

<u>Eigenspace V_3:</u> $\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} v = 0 \Rightarrow v = s\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

4b) Note: the eigenvalues for this matrix can be found by citing HW problem §7.4/2 which says that the eigenvalues of a triangular matrix are the diagonal entries.

$$p(\lambda) = \det \begin{bmatrix} (1-\lambda) & 2 & 3 \\ 0 & (2-\lambda) & 3 \\ 0 & 0 & (3-\lambda) \end{bmatrix} = (1-\lambda) \det \begin{bmatrix} 2-\lambda & 3 \\ 0 & 3-\lambda \end{bmatrix}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda) \longrightarrow \text{eigenvalues } \lambda = 1, \lambda = 2, \lambda = 3.$$

<u>Eigenspace V_1</u>: $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} v = 0 \Rightarrow v = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

<u>Eigenspace V_2</u>: $\begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} v = 0 \Rightarrow v = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

<u>Eigenspace V_3</u>: $\begin{bmatrix} -2 & 2 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} v = 0 \Rightarrow v = s \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

5a. ~~True~~ False   5b. True   5c. True

5d. True   5e. False