APPM 2360 FINAL

MAY 8, 2000

1. a. First order linear, so try integrating factor

$$y' - \frac{3}{2t}y = \frac{9}{2}t^2$$

$$p(t) = -\frac{3}{2t} \Rightarrow P(t) = \int p(t) = -\frac{3}{2}\ln t$$ (constant does not matter)

Integrating factor

$$e^{P(t)} = e^{-\frac{3}{2}\ln t} = t^{-3/2},$$

Multiplying with this gives

$$\underbrace{t^{-3/2}y' - \frac{3}{2}t^{-5/2}y}_{\frac{d}{dt}\left(t^{-3/2}y\right)} = \frac{9}{2}t^{1/2}$$

$$\Rightarrow t^{-3/2}y = 3t^{3/2} + C,$$

$$\boxed{y = 3t^3 + c \cdot t^{3/2}}$$

1b. $t^2y' + y^2 = 0$

This equation is separable - we can write it

$$\frac{y'}{y^2} = -\frac{1}{t^2}$$

Integrate:

$$-\frac{1}{y} = \frac{1}{t} + C \Rightarrow y = -\frac{t}{1+tC}$$

1c. $y'' = 2y^3$. $\quad y(0) = 1,\ y'(0) = 1.$

Solution 1:

This is of the form $y'' = F(y)$, so we can integrate it after multiplying by y':

$$y'y'' = 2y'y^3 \Rightarrow \frac{1}{2}(y')^2 = \frac{1}{2}y^4 + C.$$

Initial condition gives $\frac{1}{2}\cdot 1^2 = \frac{1}{2}\cdot 1^4 + C \Rightarrow C = 0.$

Hence: $(y')^2 = y^4 \Rightarrow y' = \pm y^2$.

Again, using initial condition, we need to choose the plus sign.

So $y' = y^2$

Separable: $\quad \frac{y'}{y^2} = 1 \Rightarrow -\frac{1}{y} = t + C,$

$$y = -\frac{1}{t+C}.$$

Initial condition $\Rightarrow 1 = -\frac{1}{0+C}$; $C = -1$

Hence $\boxed{y(t) = \frac{1}{1-t}}$.

Solution 2:

The equation is of the form $y'' = F(y, y')$.

So we set $y' = v$

and solve

$$\begin{cases} v \dfrac{dv}{dy} = 2y^3 & (1) \\ y' = v & (2). \end{cases}$$

Eq (1) can be integrated (with respect to y):

$$\frac{v^2}{2} = \frac{2y^4}{4} + c \qquad . \text{IC} \Rightarrow c = 0.$$

$$\Rightarrow \quad v^2 = y^4 .$$

Recall $v = y'$.

Hence it remains to solve $(y')^2 = y^4$.

We can now follow the same steps as in Solution 1.

2a. $x' = x^3 + 2x^2 + cx$

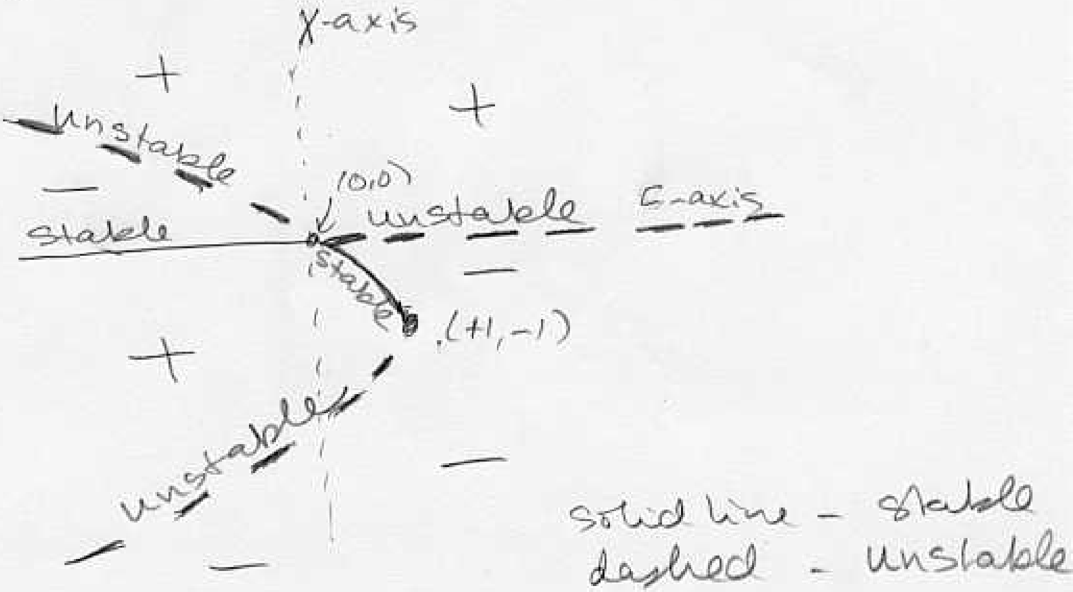
Equilibrium points occur when

$$\underbrace{x^3+2x^2+cx=0}$$

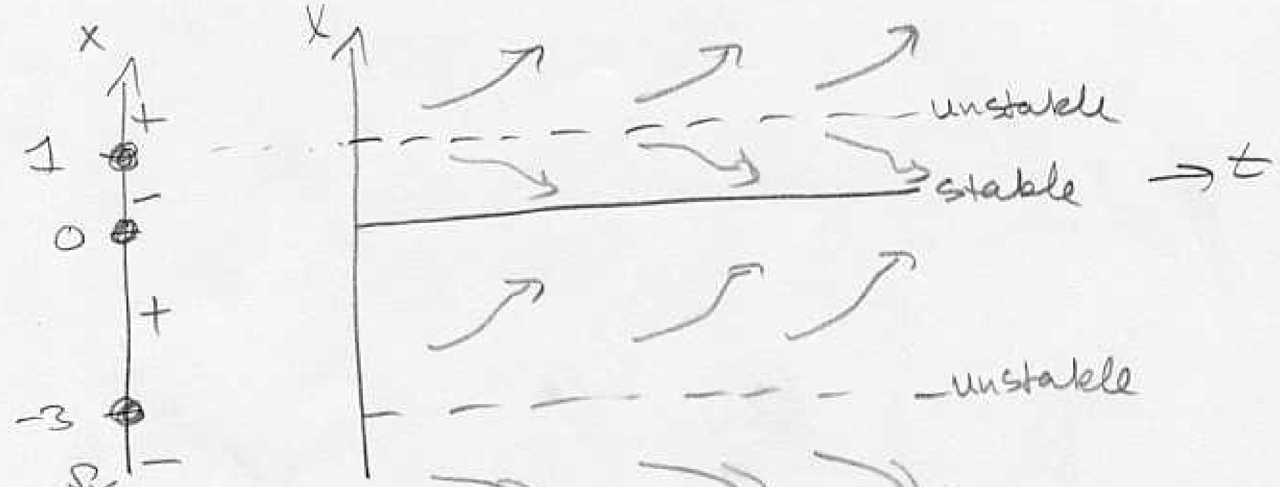
$$x(\underbrace{x^2+2x+c})$$

root $x=0$ roots of quadratic $x=-1\pm\sqrt{1-c}$.

So the bifurcation diagram becomes



2b. At $c=-3$, the equilibrium points become $-3, 0, 1$.



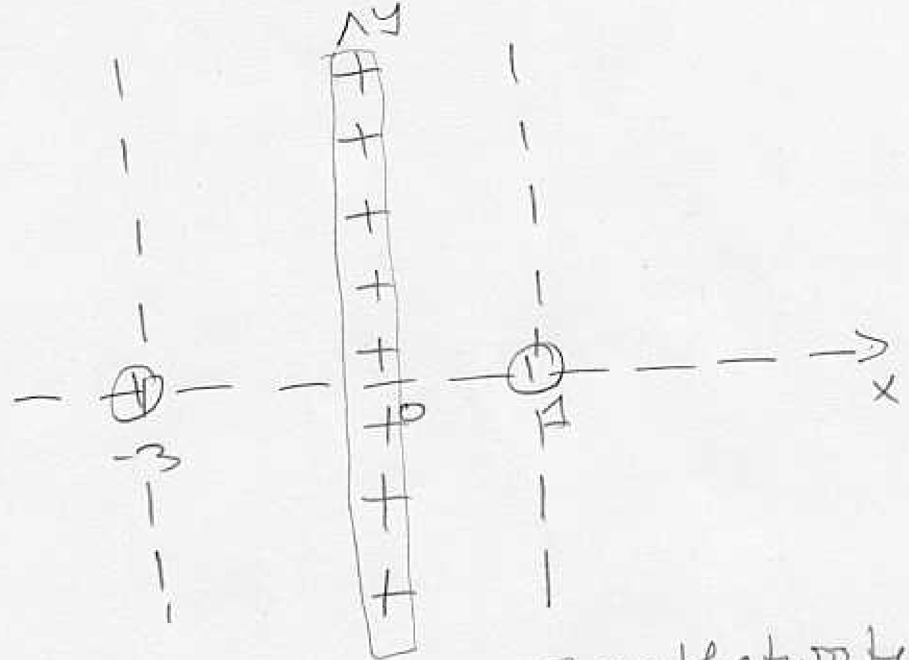
Sketch of solution curves.

2c. $\begin{cases} x' = x^3 + x^2 - 3x \\ y' = xy \end{cases}$

Previous problem (2b) gives x-nullclines at $x = -3, 0, 1$

We get also y-nullclines at $x = 0$ and $y = 0$.

In picture, below, x-nullclines are marked by direction markers "|", y-nullclines by "–"



Equilibrium points occur where the two types of nullclines intersect, i.e. $(-3,0)$, $(1,0)$ and $(0,y)$ (whole y-axis).

3. System $\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 3 & -2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Find e-values:

$$0 = \det \begin{pmatrix} 3-\lambda & 1 & 4 \\ 1 & 3-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{pmatrix} \underset{\substack{\text{expand} \\ \text{along} \\ \text{last} \\ \text{row}}}{=} (2-\lambda)\left((3-\lambda)^2 - 1\right)$$

$$\Rightarrow \lambda_1 = 4, \quad \lambda_2 = 2, \lambda_3 = 2.$$

Find e-vectors:

For $\lambda_1 = 4$:

$$\begin{pmatrix} -1 & 1 & 4 \\ 1 & -1 & -2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Last eq tells $c = 0$.
Then top eq tells $a = b$.
So $(v_1) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = 2$:

$$\begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Last eq can be omitted (No information). Top two eq. possible only if $c = 0$.
Then $a = -b$.
So $(v_2) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ No second choice possible.

Hence we need a generalized e-vector as third vector

For $\lambda_3 = 2$

$$\begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Ignore last equation, Subtract second from first

$$\begin{pmatrix} 0 & 0 & 6 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

We need $c = 1/3$, $\underbrace{a + b - \tfrac{2}{3} = -1}$

$$\Rightarrow a + b = \tfrac{-1}{3}.$$

So take for ex $|v_3\rangle = \begin{pmatrix} -1/3 \\ 0 \\ 1/3 \end{pmatrix}$

We now have

Eigenvalues: $\lambda_1 = 4$ $\quad \lambda_2 = 2$ $\quad \lambda_3 = 2$

Eig. vectors: $|v_1\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad |v_2\rangle = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\quad$ missing

Gen. Eigvector $\quad |v_3\rangle = \begin{pmatrix} -1/3 \\ 0 \\ 1/3 \end{pmatrix}$

Hence, the general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{2t} + c_3 \left\{ \begin{pmatrix} -1/3 \\ 0 \\ 1/3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\} e^{2t}$$

Next, satisfy the initial conditions

At $t=0$:

$$c_1\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}1\\-1\\0\end{pmatrix} + c_3\begin{pmatrix}-1/3\\0\\1/3\end{pmatrix} = \begin{pmatrix}2\\-1\\-1\end{pmatrix}$$

This can be written as a linear system

$$\begin{bmatrix}1 & 1 & -1/3\\ 1 & -1 & 0\\ 0 & 0 & 1/3\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}2\\-1\\-1\end{bmatrix}$$

with solution $c_1 = 0,\ c_2 = 1,\ c_3 = -3$

Finally.... the solution to ODE together with IC is

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = 0\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}e^{4t} + 1\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}e^{2t} - 3\left\{\begin{pmatrix}-\frac{1}{3}\\0\\1/3\end{pmatrix} + t\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\}e^{2t}$$

$$= \left\{\begin{pmatrix}2\\-1\\-1\end{pmatrix} + t\begin{pmatrix}-3\\3\\0\end{pmatrix}\right\}e^{2t}$$

4. $y'' = -y' + y^3 - 2y^2 - y + 2$

a, Equilibrium points; Set $y' = y'' = 0$.

$\Rightarrow \; y^3 - 2y^2 - y + 2 = 0$.

Looking ahead at part (c), we learn that $y=1$ is a root. (or we can quite easily guess 2).

Dividing out $y-1$ gives

$$y^3 - 2y^2 - y + 2 = \underbrace{(y-1)}_{\substack{\text{Root} \\ y_1 = 1}}\underbrace{(y^2 - y + 2)}_{\text{Roots } y_2 = 2,\ y_3 = -1.}$$

$\boxed{\text{So the equilibrium points are } y = -1, 1, 2}$.

b, Set $\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$

Gives

$$\boxed{\begin{cases} x_1' = x_2 \\ x_2' = x_1^3 - 2x_1^2 - x_1 + 2 - x_2 \end{cases}}$$

c. Set $\begin{cases} z_1 = y - 1 = x_1 - 1 \\ z_2 = x_2 \end{cases}$

First equation becomes

$$z_1' = z_2$$

Second equation: Plug in $x_1 = 1 + z_1$ to get

$$z_2' = x_1^3 - 2x_1^2 - x_1 + 2 - z_2 =$$

$$= 1 + 3z_1 + 3z_1^2 + z_1^3 - 2 - 4z_1 - 2z_1^2 - 1 - z_1 + 2 - z_2$$

Remove all quadratic or higher powers of z_1, simplify.

$$\Rightarrow z_2' = -2z_1 - z_2$$

So $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

d. To classify the ODE, we need to find the eigenvalues.

$$0 = \det \begin{pmatrix} 0-\lambda & 1 \\ -2 & -1-\lambda \end{pmatrix} = \lambda^2 + \lambda + 2$$

$$\Leftrightarrow \lambda_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1-8}{4}} = \frac{-1 \pm i\sqrt{7}}{2}.$$

The eigenvalues are complex and with a negative real part. So the solution is a Spiral (going in).

e. Homogeneous (undriven) equation

$$z'' + z' + 2z = 0$$

Characteristic equation

$$r^2 + r + 2 = 0$$

$$r_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1-8}{4}} = \frac{-1 \pm i\sqrt{7}}{2}$$

$$y_h = c_1 e^{\frac{-1+i\sqrt{7}}{2}t} + c_2 e^{\frac{-1-i\sqrt{7}}{2}t}$$

$$\left(= D_1 e^{-t/2} \cos \frac{\sqrt{7}}{2} t + D_2 e^{-t/2} \sin \frac{\sqrt{7}}{2} t \right)$$

Now we need to find one particular solution to the driven equation:

Let the forcing function be $e^{(1+i)t}$; then take imaginary part at end.

Write the ODE as

$$P(D)[u] = e^{(-1+i)t}$$

(D^2+D+2)

Assume form $h(t)\, e^{(-1+i)t}$

By formula, we get

$$\cancel{e^{(-1+i)t}}\, \underbrace{P(D-1+i)}\,[h] = \cancel{e^{(-1+i)t}}$$

$$\left[D^2-(1-2i)D + (1-i)\right][h] = 1$$

So we can choose h as a constant:

$$h = \frac{1}{1-i} = \frac{1+i}{2}$$

Then $u = e^{-t}(\cos t + i \sin t)\cdot \frac{1}{2}(1+i)$

Imaginary part: $\dfrac{e^{-t}}{2}(\cos t + \sin t)$

Finally:

$$\boxed{u = D_1 e^{-t/2}\cos\frac{\sqrt{7}}{2}t + D_2 e^{-t/2}\cos\frac{\sqrt{7}}{2}t + \frac{e^{-t}}{2}(\cos t + \sin t)}$$

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5 a. FALSE — solution is $y(t) = \tan t$, blows to $+\infty$ at $t = \pi/2$

b. FALSE — Solution to homogeneous part grows exponentially

c. TRUE

d. FALSE — $\sin 2x$ repeats after x increases by π,
$\cos 3x$ — " — $\frac{2\pi}{3}$.
So sum will repeat after time 2π.

e. FALSE — Holds only if $f_1 = f_2$.

f. FALSE — This is a resonance case

g. TRUE

h. FALSE — True only for odd orders.