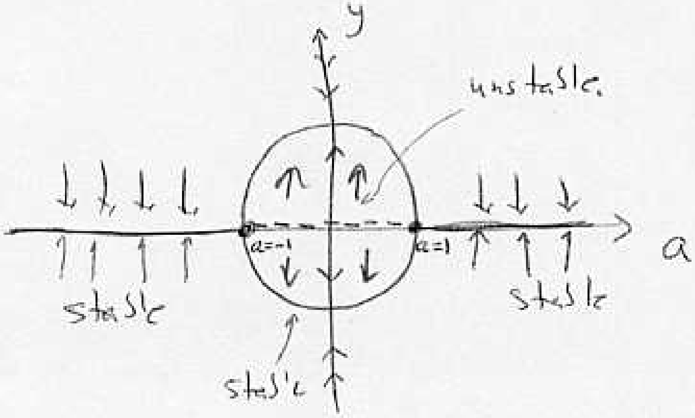
1)

a) Equilibrium solutions: $y' = 0$

$$y(1-a^2-y^2) = 0$$

$y_1 = 0$ or $a^2+y^2=1 \Rightarrow$ bifurcation points $a = \pm 1$

b)



2) First, solve undriven ode $y'' - 5y' + 6y = 0$

$\Rightarrow \quad r^2 - 5r + 6 = 0 \quad \Rightarrow \quad (r-2)(r-3) = 0$

$r_1 = 2$ and $r_2 = 3$.

$$y_u(t) = c_1 e^{2t} + c_2 e^{3t}.$$

a) guess $f(t) = 1 \quad y_d = A \Rightarrow \quad 6A = 1 \Rightarrow A = \frac{1}{6}$.

$\Rightarrow \quad \boxed{y(t) = c_1 e^{2t} + c_2 e^{3t} + \frac{1}{6}.}$

b) $y_d = Bte^{2t} \Rightarrow \quad y_d' = B(1+2t)e^{2t}$

$y_d'' = B(2+2+4t)e^{2t} = B(4+4t)e^{2t}$

2)

$$B(4+4t)e^{2t} - 5B(1+2t)e^{2t} + 6Bte^{2t} = 2e^{2t}$$

$$-Be^{2t} + \cancel{4Bte^{2t}} - \cancel{10Bte^{2t}} + \cancel{6Bte^{2t}} = 2e^{2t}$$

$$\Rightarrow \quad \boxed{B=-2}$$

$$\boxed{y(t) = c_1 e^{2t} + c_2 e^{3t} \quad -2te^{2t}}$$

c) write $\quad y_d(t) = A\cos(2t) + B\sin(2t)$

$$y_d' = -2A\sin(2t) + 2B\cos(2t)$$

$$y_d'' = -4A\cos(2t) - 4B\sin(2t)$$

$\Rightarrow$

$$-4A\cos(2t) - 4B\sin(2t) + 10A\sin(2t) - 10B\cos(2t)$$
$$+ 6A\cos(2t) + 6B\sin(2t) = 2\cos 2t.$$

$\Rightarrow$

$$(-4A - 10B + 6A)\cos 2t + (-4B + 10A + 6B)\sin 2t$$
$$= 2\cos 2t$$

$\Rightarrow$

$$2A - 10B = 2$$
$$2B + 10A = 0 \Rightarrow \quad \boxed{B = -5A}$$

$$2A + 50A = 2 \Rightarrow \quad 52A = 2 \Rightarrow \quad \boxed{A = \frac{1}{26}}$$

and $\quad \boxed{B = -\frac{5}{26}}$

(3)

$$y_d(t) = \frac{1}{6} + \frac{1}{26}\cos(2t) - \frac{5}{26}\sin(2t)$$

$$\Rightarrow \boxed{y(t) = c_1 e^{2t} + c_2 e^{3t} + \frac{1}{6} + \frac{1}{26}\cos(2t) - \frac{5}{26}\sin(2t)}$$

(3)

$$\theta'' + 3\theta' + 2\theta = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0 \Rightarrow r_1 = -1, \quad r_2 = -2.$$

$$\Rightarrow \quad \theta(t) = c_1 e^{-2t} + c_2 e^{-t}$$

$$\theta'(t) = -2c_1 e^{-2t} - c_2 e^{-t}$$

$$\Rightarrow \quad \theta'(0) = -2c_1 - c_2 = 0 \Rightarrow \quad \boxed{c_2 = -2c_1}$$

$$\boxed{\theta(t) = c_1 e^{-2t} - 2c_1 e^{-t}}$$

$$\theta(0) = c_1 - 2c_1 = \theta_0 \Rightarrow \quad \boxed{c_1 = -\theta_0}$$

$$\boxed{\theta(t) = \theta_0\left(2e^{-t} - e^{-2t}\right)}$$

(4)

$$\theta(t) = \frac{1}{2}\theta_0$$

$$\Rightarrow \quad 2e^{-t} - e^{-2t} = \frac{1}{2}$$

$$e^{-t} = x$$

$$2x - x^2 = \frac{1}{2}$$

$$2x^2 - 4x + 1 = 0.$$

$$x = \frac{4 \pm \sqrt{16-8}}{4} = \frac{4 \pm \sqrt{8}}{4}$$

$$x = \frac{4 \pm 2\sqrt{2}}{4} = 1 \pm \frac{\sqrt{2}}{2}$$

now $e^{-t} = x \Rightarrow \quad t = -\ln x$

$$\Rightarrow \quad \boxed{t = -\ln\left(1 - \frac{\sqrt{2}}{2}\right)}$$

Need to choose the negative sign here - else t would become negative.

b. If instead

$$\theta(t) = \frac{1}{4}\theta_0$$

$$\Rightarrow \quad 2x - x^2 = \frac{1}{4}$$

$$x = \frac{2 \pm \sqrt{3}}{2}$$

$$e^{-t} = x \Rightarrow t = -\ln x$$

$$\Rightarrow \boxed{t = -\ln\left(1 - \frac{\sqrt{3}}{2}\right)}$$

Need again to choose negative sign.

Is this time twice the previous time?

$$-2\ln\left(1 - \frac{\sqrt{2}}{2}\right) \stackrel{?}{=} -\ln\left(1 - \frac{\sqrt{3}}{2}\right)$$

Simplify

$$\ln\left(1 - \frac{\sqrt{2}}{2}\right)^2 \stackrel{?}{=} \ln\left(1 - \frac{\sqrt{3}}{2}\right);$$

$$1 \stackrel{?}{=} 2\sqrt{2} - \sqrt{3}.$$

Take square etc $\Rightarrow$ $24 \stackrel{?}{=} 25$. <u>NO</u>

4.

	i	ii	iii	iv
4a			X	
b			X	
c			X	
d			X	
e	X			

Alternate solution to 2c:

To solve $y'' - 5y' + 6y = 2\cos 2t$,

Consider: $z'' - 5z' + 6z = 2e^{2it}$

(Then take $y_p = \text{Real}(z_p)$)

Solve for ex. by guessing $z_p = Ae^{2it}$

Plug in:

$$\underbrace{(-4 - 5\cdot 2i + 6)}_{(2-10i)} A \cdot \cancel{e^{2it}} = 2\cancel{e^{2it}}$$

$$A = \frac{2}{2-10i} = \frac{1}{1-5i} = \frac{1+5i}{1+25} = \frac{1}{26}(1+5i)$$

$$z_p = \frac{1}{26}(1+5i)(\cos 2t + i\sin 2t) =$$

$$= \frac{1}{26}(\cos t - 5\sin t) + \frac{i}{26}(5\cos 2t + \sin 2t)$$

Take real part!

So $y_p = \frac{1}{26}(\cos t - 5\sin t)$.

Finally, $f(t) = 1 + 2\cos 2t$, So we have to add the solution of part (a):

Answer to 2c: $y(t) = C_1 e^{2t} + C_2 e^{3t} + \frac{1}{6} + \frac{1}{26}\cos(2t) - \frac{5}{26}\sin(2t)$