Solution to Exam 3 Fall 2000.

1. The system will have more solutions when $\det(A) = 0$ (it will then have either 0 or ∞ many solutions - and we know it has at least one solution: $\begin{pmatrix}0\\0\\0\end{pmatrix}$. So it must then have infinitely many).

$$0 = \det(A) = \det\begin{bmatrix} 1 & -1 & a \\ 3 & -3 & 1 \\ a & -1 & 2 \end{bmatrix} = \begin{pmatrix}\text{add col 1} \\ \text{to col 2}\end{pmatrix}$$

$$= \det\begin{bmatrix} 1 & 0 & a \\ 3 & 0 & 1 \\ a & a-1 & 2 \end{bmatrix} = \begin{pmatrix}\text{expand along} \\ \text{col 2}\end{pmatrix}$$

$$= -(a-1)\det\begin{bmatrix} 1 & a \\ 3 & 1 \end{bmatrix} = -(a-1)(1-3a).$$

This is zero when $\boxed{a = 1 \text{ or } a = 1/3}$.

2a.

$$p(\lambda) = \det\begin{bmatrix} 2-\lambda & 3 & 0 \\ 2 & 1-\lambda & -1 \\ 2 & 6 & 1-\lambda \end{bmatrix} = \quad \text{(expand along column 3)}$$

$$= 1\cdot\det\underbrace{\begin{bmatrix} 2-\lambda & 3 \\ 2 & 6 \end{bmatrix}}_{\left((2-\lambda)\cdot 6 - 2\cdot 3 = \; = -6\lambda+6\right)} + (1-\lambda)\det\underbrace{\begin{bmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix}}_{\left((2-\lambda)(1-\lambda) - 2\cdot 3 = \; = \lambda^2-3\lambda-4\right)} =$$

$$= 6(1-\lambda) + (\lambda^2-3\lambda-4)(1-\lambda) =$$

$$= (1-\lambda)\underbrace{(\lambda^2-3\lambda+2)}_{(\lambda-1)(\lambda-2)} = -(\lambda-1)^2(\lambda-2).$$

So eigenvalues are $\boxed{\lambda_1 = 2,\ \lambda_2 = \lambda_3 = 1.}$

If one obtains $p(\lambda) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$, one needs to find a root by inspection, say $\lambda = 1$. Then divide out $(\lambda - 1)$ to get

$P(\lambda) = -(\lambda-1)(\lambda^2 - 3\lambda + 2)$ etc.

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b. $tr(A) = \sum a_{ii} = a_{11} + a_{22} + a_{33} = 2+1+1 = 4$

$\sum \lambda_i = \lambda_1 + \lambda_2 + \lambda_3 = 2+1+1 = 4.$

So checks ok

c.

$\lambda_1 = 2$

$$\left[\begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ 2 & 6 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ 0 & 7 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 7 & 0 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\left.\begin{array}{l} v_3 = s, \\ v_2 + 0 \cdot s = 0 \Rightarrow v_2 = 0 \\ 2v_1 - 0 - s = 0 \Rightarrow v_1 = s/2 \end{array}\right\} \Rightarrow v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}$$

Eigenvector

$\lambda_2 = \lambda_3 = 1$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 2 & 6 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 6 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -6 & -1 & 0 \\ 0 & 6 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} v_3 = s \\ 6v_2 + s = 0 \Rightarrow v_2 = -s/6 \\ v_1 + 3(-s/6) = 0 \Rightarrow v_1 = s/2 \end{array}\right\} \Rightarrow v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1/2 \\ -1/6 \\ 1 \end{pmatrix}$$

One eigenvector only

Generalized e-vector:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 1/2 \\ 2 & 0 & -1 & -1/6 \\ 2 & 6 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1/2 \\ 2 & 0 & -1 & -1/6 \\ 0 & 6 & 1 & 7/6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1/2 \\ 0 & -6 & -1 & -7/6 \\ 0 & 6 & 1 & 7/6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1/2 \\ 0 & 6 & 1 & 7/6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} v_3 = s \\ 6v_2 + s = 7/6, \quad v_2 = 7/36 - s/6 \\ v_1 + 3\left(\frac{7}{36} - \frac{s}{6}\right) = \frac{1}{2} \Rightarrow v_1 = -\frac{1}{12} + s/2 \end{array} \Rightarrow v = \begin{pmatrix} -\frac{1}{12} \\ 7/36 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1/2 \\ -1/6 \\ 1 \end{pmatrix}$$

Generalized e-vector

Confirms previous e-vector

3. $\begin{pmatrix} x' \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x \end{pmatrix} \quad x(0) = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

$p(\lambda) = \det \begin{vmatrix} 3-\lambda & 8 \\ -1 & -3-\lambda \end{vmatrix} = -(3-\lambda)(3+\lambda) + 8 = -(9-\lambda^2) + 8 = \lambda^2 - 1$

$\lambda_1 = 1, \ \lambda_2 = -1$

$\underline{\underline{\lambda_1 = 1}} \quad \begin{pmatrix} 2 & 8 & 0 \\ -1 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad v = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$\lambda_2 = -1 \quad \begin{pmatrix} 4 & 8 & 0 \\ -1 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad w = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

General solution $x_{gen}(t) = c_1 e^{t}\begin{pmatrix} -4 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\underline{\underline{\text{IVP}}}$: $\begin{pmatrix} -4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

$\begin{pmatrix} -4 & -2 & 6 \\ 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 & -2 \\ 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix}$

$c_2 = -1 \qquad c_1 + c_2 = -2 \Rightarrow c_1 = -2 + 1 = -1$

IVP solution: $\boxed{x(t) = (-1)e^{t}\begin{pmatrix} -4 \\ 1 \end{pmatrix} + (-1)e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}}$

4	i	ii	iii	iv
a	X			
b				X
c			X	
d			X	
e			X	

a) $\begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = \lambda^2 - 2a\lambda + (a^2-b^2)$ $\lambda = \dfrac{2a \pm \sqrt{4a^2-4(a^2-b^2)}}{2}$

$\lambda = a \pm b$

b) $x_{gen}(t) = c_1 e^{-3t}(u) + c_2 e^{-2t}(v) + c_3(w)$

$x_{gen}(t) \not\to \infty$ as $t \to \infty$

c) cooperation (from positive terms $3xy$ and $2xy$).

d) $\det(AB) = \det A \det B = \det B \det A = \det(BA)$

e) $x_1' = x_1 - x_2 - t^2 \Rightarrow x_2 = x_1 - t^2 - x_1'$

$x_1'' = x_1' - x_2' - 2t$

$x_2' = x_1 + 3x_2 - 2t$

$x_1'' = x_1' - x_1 - 3x_2 + 2t - 2t$

$x_1'' = x_1' - x_1 - 3(x_1 - t^2 - x_1')$

$x_1'' = 4x_1' - 4x_1 + 3t^2$

$x_1'' - 4x_1' + 4x_1 = 3t^2$