FINAL 2360 Dec 20, 2000

SOLUTION SET

1. a. $y' = e^{t+y} = e^t \cdot e^y$

$e^{-y} \cdot y' = e^t$, $\int e^{-y} dy = \int e^t dt$. $-e^{-y} = e^t + c$

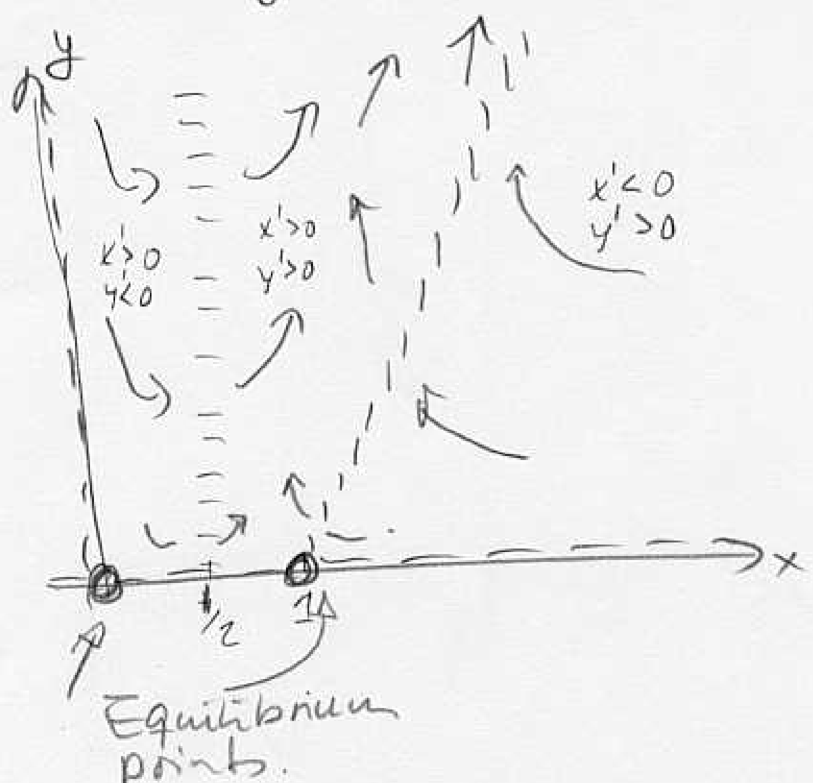
$\boxed{y = \ln(-e^t - c)}$

b. $y' - y = 5e^t$. $p(t) = -1$, $P(t) = -t$, Integrating factor e^{-t}.

$\underbrace{e^{-t} y' - e^{-t} y}_{\frac{d}{dt}(e^{-t}y)} = 5$ $\Rightarrow e^{-t} y = 5t + c$, $\boxed{y = 5te^t + ce^t}$

2a. $\begin{cases} x' = (3 - 3x + y)x \\ y' = (-2 + 4x)y \end{cases}$ x nullcline $\{x = 0\}, \{y = 3x - 3\}$ mark $\;'\;'\;$

y nullcline $\{y = 0\}, \{x = \frac{1}{2}\}$ $---$



b, For any starting point away from the axes (i.e. with both species present), population of both grow to infinity.

(Starting with one species only, $y \to 0$ or $x \to 1$).

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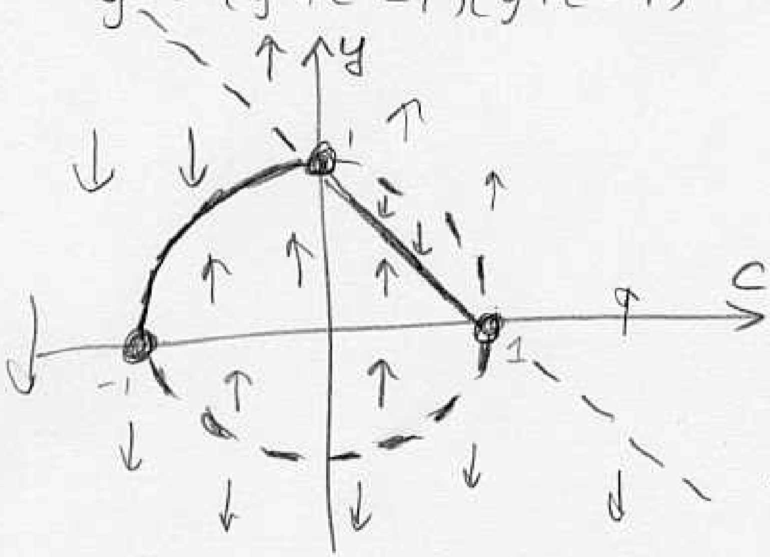
3. $y' = (y^2+c^2-1)(y+c-1)$ $\qquad$ $y'=0 \Rightarrow \begin{cases} y^2+c^2=1 \\ \text{or} \\ y=-c+1 \end{cases}$.

Circle and straight line.



Stable equilibria solid, unstable dashed.

Three equilibrium points marked.

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4 a. Given $\quad P(D)[y] = (D-1)(D+2)[y] = f(t)\cdot e^t$

Plug in $y = h\cdot e^t$

$$\underbrace{P(D)[h\cdot e^t]}_{\text{By formula}} = f(t)\cdot e^t$$

$$e^t P(D+1)[h] = f(t)e^t$$

Now cancel e^t, left with

$$((D+1)-1)((D+1)+2)[h] = f$$

$$D(D+3)[h] = f \text{ , i.e. } \boxed{h'' + 3h = f}$$

4b. $y'' = \frac{8(y-1)}{(y')^2}$

Equation of form $y'' = F(y, y')$.

So set $y' = v(y)$.

Then $y'' = \frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = v\frac{dv}{dy}$.

ODE $\Rightarrow$

$v\frac{dv}{dy} = \frac{8(y-1)}{v^2}$, $\int v^3 dv = \int 8(y-1)dy$

$\frac{v^4}{4} = 8\left(\frac{y^2}{2} - y\right) + C_1$

From initial cond: $y(0) = 1$, $v(0) = 0$ So

$0 = 8\left(\frac{1}{2} - 1\right) + C_1$; $C_1 = 4$,

$\frac{v^4}{4} = 4y^2 - 8y + 4 = 4(y-1)^2$

$v^2 = 4(y-1)$; $\underset{\frac{dy}{dt}}{\underbrace{v}} = \pm 2\sqrt{y-1}$.

$\Rightarrow 2\,dt = \pm\frac{dy}{\sqrt{y-1}}$, $2t = \pm 2\sqrt{y-1} + C_2$

$y(0) = 1 \Rightarrow 0 = 0 + C_2$, $C_2 = 0$.

$\Rightarrow 2t = \pm 2\sqrt{y-1} \Rightarrow t^2 = y - 1$,

$\boxed{y = t^2 + 1}$.

5. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

a. Triangular matrix, so eigenvalues in diagonal.

$\lambda_1 = \lambda_2 = \lambda_3 = 1$.

Find eigenvector(s):

$|A - \lambda I||v\rangle = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$: $\left[\begin{array}{ccc:c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Now upper triangular.

$v_3 = s$

$1 \cdot v_2 + 0 \cdot s = 0 \Rightarrow v_2 = 0$

$0 \cdot v_1 + 0 \cdot 0 + 0 \cdot s = 0 \Rightarrow v_1 = t$

$|v\rangle = \begin{pmatrix} t \\ 0 \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

two eigenvectors $|v_1\rangle, |v_2\rangle$

Need to look for a generalized e-vector (not sure yet if start chain from $|v_1\rangle$ or $|v_2\rangle$.

Try $|v_1\rangle$:

$\left[\begin{array}{ccc:c} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$ Top equation clearly impossible.

Try $|v_2\rangle$

$\left[\begin{array}{ccc:c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc:c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$0 \cdot v_3 = 0$, $v_3 = s$

$1 \cdot v_2 + 0 \cdot s = 1$, $v_2 = 1$

$0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 = 0$, $v_1 = t$

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$$\begin{pmatrix} v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Generalized e-vector

Confirms two eigenvectors.

Hence:

E-val.	$\lambda_1 = 1$	$\lambda_2 = 1$	$\lambda_3 = 1$
E-vect:	$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	—
Gen e-vect:	—	—	$v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

chain.

b. General ODE solution

$$\begin{pmatrix} x \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 e^t \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

6a. $\left.\begin{array}{l} (D^2+1)(\sin t + \cos t) = 0 \\ (D^3)(t^2) = 0 \end{array}\right\}$ Degree 5 $\Rightarrow$ ii

b. Char eq and e-val:

(i) $\lambda^2+9\lambda+18=0$ $\lambda_1=-3, \lambda_2=-6$ improper node arrows going in

(ii) $\lambda^2-2\lambda-8=0$ $\lambda_1=-2, \lambda_2=+4$ saddle

$\boxed{\text{(iii) } \lambda^2-18\lambda+56=0 \quad \lambda_1=4,\ \lambda_2=14 \quad \text{improper node. arrows out.} \quad \left(\text{E-vect } \begin{pmatrix}-2\\1\end{pmatrix},\ \begin{pmatrix}4\\3\end{pmatrix} \text{ \& direction of primary axes} \right)}$

(iv) $\lambda^2+2\lambda+2$ $\lambda_1=-1+i, \lambda_2=-1-i$ spiral in

Answer (iii)

c. $y''+y=\cos t$ features resonance: (i)

$y_h = \underbrace{c_1\cos t}_{\text{matches RHS.}} + c_2\sin t.$

d. $y_0=1$

$y_1 = 1+\frac{1}{2}y'(0)=1$ (ii)

$y_2 = 1+\frac{1}{2}y'\big|_{t=\frac{1}{2},y=1} = 1+\frac{1}{4}=5/4$

e. $P(D)y_1=\sin t$ Mult by 3 } Add $\Rightarrow$

$P(D)y_2=e^t$ Mult by 1 }

$P(D)\underbrace{[3y_1+y_2]}_{y}=3\sin t+e^t$ (ii)